The Mathematics Teacher

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DEPARTMENTS

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The ancients versus the moderns, a new battle of the books[1]

MORRIS KLINE, *New York University, New York, New York.*

Teachers of mathematics must face the issues pertaining to the reform of the high school curriculum. Morris Kline clearly and forcefully expresses one point of view. He has been kind enough to consent to an arrangement which enabled A. E. Meder, Jr., Rutgers University, to formulate a reply published on pages 428–433 of this issue.

INTRODUCTION

BEFORE I ENTER upon my discussion of curriculum, I should like to take this occasion to express my thanks to Professor Howard Fehr for the opportunity to speak to you today. Professor Fehr and I have delivered talks at other meetings and, at least insofar as he has represented the Commission on Mathematics of the College Entrance Examination Board, he and I have taken different positions. Nevertheless he invited me to speak because he wanted all points of view to be presented to you. I regard his invitation therefore as proof of his high-mindedness, liberality, and true leadership of The National Council of Teachers of Mathematics.

I have one more supplementary remark to make. The major part of my talk is a critique of the movement to teach modern mathematics in the high schools. My understanding of what this movement amounts to has been derived from talks I have heard, from conversations with numerous advocates of the movement, from printed materials, and from articles appearing in THE MATHEMATICS TEACHER and other journals. In particular, there is a rather definitive article written by Dean A. E. Meder in THE MATHEMATICS TEACHER for October 1957. This article appeared to be so important in presenting the views of the Commission on Mathematics that tens of thousands of preprints were run off and distributed around the country. But just a month ago, Dean Meder and I debated with each other at Ohio State University, and Dean Meder presented previews of the Commission's forthcoming recommendations that were more moderate than those advocated in the published article. I am not at all satisfied with the more moderate views either, and I shall try to show why a little later. However, the moderation may indicate that perhaps insofar as the Commission is concerned, modern mathematics is dying. But I recall that it took thirty stabs to kill Caesar. Moreover, if this hydra called modern mathematics has lost one of its heads, there are others to be chopped off. I shall proceed therefore to do the dirty work.

CRITICISM THAT TRADITIONAL MATHEMATICS IS OUTMODED

My first objection is perhaps more of a quarrel about words rather than about substance, about arguments presented by

[1] This material was originally presented at the Thirty-sixth Annual Meeting of the National Council of Teachers of Mathematics at Cleveland, Ohio on April 11, 1958.

the modernists rather than about the subject matter they are proposing. Nevertheless I believe that the point I shall make warrants some attention. The argument most often offered by the modernists is that the mathematics we have been teach- is outmoded because it is several hundred years old. This argument impresses people because we are prone in our modern civilization to believe that what is new is automatically better than what is old. To people who believe that a 1958 automobile is already out of date by the fall of 1958 it seems immediately clear that three-hundred-year-old mathematics must be hopelessly antiquated. Leaving aside the question of whether a 1958 automobile is really better than its predecessors, let me point out that if there is one subject in which up-to-dateness is totally irrelevant, it is mathematics. Mathematics has been built solidly, at least since Greek times, and generally the later developments have depended logically upon the earlier ones. Mathematics is, in other words, a cumulative development. Hence the older portions do not become antiquated or useless. In a subject such as physics, a new theory will often entirely replace an old one. For example, we no longer speak of a phlogiston theory of combustion. But this is not true of mathematics. The geometry of Euclid and the trigonometry of Hipparchus and Ptolemy, although two thousand years old, are still not replaced by a new geometry and trigonometry that show the old subjects to be incorrect or useless.

Moreover, the modernists themselves do not wish to throw out all of the older mathematics. They would teach some Euclid and some trigonometry. They certainly wish to teach the calculus and coordinate geometry that are three hundred years old. If the argument that what is old is definitely bad is a sound one, then certainly all of the old mathematics should be thrown out.

Now I know that the modernists don't mean to argue that what is old is necessarily bad. Hence all that I ask is that the modernists stop emphasizing the word "modern" with all its connotations. That word serves only as propaganda or as an appeal to an argument by analogy, and every mathematician knows that an argument by analogy is not reliable.

The modernists also give the impression that the topics they recommend are the main ones now claiming the attention of mathematicians. But many more mathematicians are still advancing such subjects as ordinary and partial differential equations, integral equations, numerical analysis, algebraic geometry, integral transform theory, approximation procedures, and other branches of analysis. These modern mathematical interests and activities somehow have escaped mention.

Value of modern mathematics for mathematics proper

Let us put aside, however, any contentions about a weak or misdirected argument and examine on its own merits the contents which the modernists advocate. The modernists would replace material currently taught by such topics as symbolic logic, Boolean algebra, set theory, some topics of abstract algebra such as groups and fields, topology, postulational systems, and statistics. I have no objection to the introduction of statistics, but I should like to examine the value of the other subjects from the standpoint of their central position in the body of mathematics and from the standpoint of application.

As for importance in the body of mathematics, I believe it is fair to say that one could not pick more peripheral material. Let me consider a few of these subjects, for example, symbolic logic. No mathematician except a specialist in problems of mathematical logic uses symbolic logic. First of all, every mathematician thinks intuitively and then presents his arguments in a deductive form, using words, familiar mathematical symbols, and common logic. This is true of 99.9 per cent of all the mathematics that has been created. Specialists in foundation problems, who have

to worry about the ambiguities and imprecision of ordinary language, do resort to symbolic logic. But even these people know what they want to say intuitively and *then* express their thoughts in special symbols. Let us be clear that the symbolic logic does not control or direct the thinking; it is merely the compact written expression of the real thinking. One must know what he wants to say before he puts it in symbolic form and, indeed, make sure that the symbols express what is intended rather than that the symbols tell him what he means to say. Hence not only is symbolic logic *not* used by almost all mathematicians, but those who do use it do their effective thinking first in more common language. Moreover, if the investigations in symbolic logic have taught us anything, it is that this medium cannot encompass any significant portion of mathematics.

Let us consider next the centrality of topology in mathematics. Topology is a new and as yet relatively untried branch of mathematics. It makes no contribution to the essential subject matter of algebra, trigonometry, co-ordinate geometry, calculus, and differential equations. As to its role in geometry, Euclidean geometry, non-Euclidean geometry, projective geometry, and differential geometry are far more basic. Let me also add a caution here. Such topics as the Königsberg bridge problem, the four-color problem, and the Möbius band are not topology. They are incidental details, curiosities, and amusements of the caliber of magic squares. The real matter of topology has not even been mentioned by the modernists; hence they should not deceive themselves and others that they are teaching topology.

Boolean algebra plays no role at all in mathematics. Set theory, which is closely related, is used only in topology and in the foundations of the theory of real functions and of the theory of probability. Set theory, then, plays a limited role in mathematics.

Insofar as elementary mathematics is concerned the concepts of abstract algebra are of use only in unifying some fundamental branches. After one knows the various types of real numbers, complex numbers, vectors, matrices, and transformations, it is nice to learn that these many mathematical entities have a few properties in common. But if one does not know, and know thoroughly, these basic subjects, then the group concept, for example, has nothing to offer. It appears to be an empty subject, an arbitrary creation of mathematical phantasy.

The Value of Modern Mathematics for Applications

And now let me consider the second value claimed for these modern topics, namely, their use today in applications. My evaluation is that they are practically useless. Some examples have been offered to show how Boolean algebra and symbolic logic can be used to form committees. But these committees will do less good than some of our Congressional committees. Of course such applications are totally artificial. The claim is made that Boolean algebra is used to design switching circuits. This claim is technically correct, but it justifies teaching Boolean algebra to high school students about as much as the argument that the Presidency of the United States is important because it publicizes the game of golf. The claim is made that the newer mathematics is wanted by the social scientists. Here we must face some hard facts. Except through statistics, the social sciences have made no real applications of mathematics and, in fact, are still getting nowhere fast. The social sciences are hardly sciences, let alone mathematical sciences. If fifty or one hundred years from now the social sciences make some significant use of mathematics, any kind of mathematics, we'll teach it. But even then there will be some question as to whether it should be taught on the high school level.

It seems curious to me that when the modernists talk about applications they make no mention of the application of

mathematics to mechanics, sound, light, radio, electricity, atomic and nuclear theory, hydrodynamics, aerodynamics, geophysics, magnetohydrodynamics, elasticity, plasticity, chemistry, chemical physics, physical biology, and the various branches of engineering, If there are ten people in this world designing switching circuits, there are a hundred thousand scientists in these other fields. And what mathematics is used in these fields I have mentioned? Because I work in a large institute devoted to applied mathematics, I believe I can tell you. The subjects are algebra, geometry, trigonometry, co-ordinate geometry, the calculus, ordinary and partial differential equations, series, the calculus of variations, differential geometry, integral equations, theory of operators, and many other branches of analysis.

The efficiency of the abstract approach to traditional mathematics

The modernists often claim that they wish to teach much of the standard material but that they will do so more efficiently through a unified point of view made possible by the modern concepts. The prime example they offer is the concept of a field, and the presumed advantage of teaching fields first is that the rational, real, and complex numbers form a field. Setting aside for the moment the difficulties young people may have in learning the concept of a field, let us see what is gained. When one has a field, one has a system of elements that combine under two operations. These operations satisfy five properties, closure, the associative and commutative laws, and the existence of an identity and of an inverse. In addition, there is the distributive law connecting the two operations. Suppose a student knew all these properties. Would he be able to add 2 and 2? Would he be able to add fractions, irrationals, complex numbers? No. And the reason is that just because the field properties are those which are *common* to all these systems of numbers, they automatically wipe out any distinguishing features. The field concept wipes out just those important processes which are needed to operate with these basic number systems. I conclude that a person with an excellent knowledge of fields could not even make change in a grocery store. To put it otherwise, fields do not explain the types of numbers and operations with them. In fact, it is just the reverse. A good understanding of the various number systems explains the concept of a field. Hence insofar as efficiency is concerned, the time that is wasted is the time spent teaching the abstract concept.

The same kind of argument applies to a subject such as topology. Even if we really taught topology and not the curiosities commonly described as topology, the students could not with this knowledge find the area of a triangle, because by very definition topology is not concerned with particular metric geometries. We would still have to teach the specialized geometry.

Let us remember that the more general the mathematical concept, the emptier it is.

Teaching the abstract before the concrete

The modernists make other claims for their material. They say that their material is a new point of view as much as new content. This is true, but what is the point of view? In the first place, *their* modern mathematics is highly abstract. Groups, rings, and fields, as I have already indicated, are abstractions from various more concrete algebras. Topology is a generalization of Euclidean, non-Euclidean, and projective geometry. Postulational systems are an abstraction from the deductive structure of the various branches of mathematics. Even the simple concepts have become abstract. The old-fashioned x which used to stand for an unknown number is now the place holder for the pronumeral of the name of a number, or is it the pronumeral of the place holder of the number of names? Of numbers them-

selves we can no longer speak. We have only names for numbers. The numbers themselves are in heaven.

How can we expect students to learn the abstract before the concrete? I thought that the educators were the ones who always stressed that the concrete must come first. Certainly if one judges by the time it took great mathematicians to learn the concrete cases, one can hardly believe that the abstract ideas can be taught to young people. It took mathematicians a few thousand years to understand the irrational number, and yet we presumably can teach Dedekind cuts to high school people. It took mathematicians three hundred years to understand complex numbers, but we can teach at once that a complex number is an ordered pair of real numbers. It took about a thousand years to understand negative numbers, but now we have only to say that a negative number is an ordered pair of natural numbers and, presto, the idea is learned. From Galileo to Dirichlet, mathematicians struggled to understand the concept of a function, but now domain, range, and ordered pairs such that no two second members may be different for the same first member do the trick. From the ancient Egyptians and Babylonians to Vieta and Descartes, no mathematician realized that letters could be used to stand for a class of numbers, but now we are told that the simple notion of set yields the concept immediately. The point of an investigation such as topology came to mathematicians after they had analyzed Euclidean and the other geometries I have mentioned, but somehow these other geometries are apparently not needed to see the point of topology.

Were these great mathematicians really not so great, or are there not intrinsic difficulties in all these concepts that cannot be hurdled by jumping into abstractions?

Criticism of the Modernist Rigor

The second major change in viewpoint which the modernists advocate is more rigor. We can no longer present Euclidean geometry à la Euclid but must present it à la Hilbert. Dedekind cuts and Peano's axioms are the new order of the day in teaching numbers. And symbolic logic must rectify the weaknesses of Aristotelian logic.

The desire to incorporate more rigor is ill-advised for many reasons. First of all, the capacity to appreciate rigor must be developed. The capacity to appreciate rigor is a function of the age of the student and not of the age of mathematics. Those who do not believe that mathematical maturity is needed to appreciate rigor should be guided in this matter, too, by the history of mathematics. For two thousand years after it was created, the best minds in Western civilization accepted Euclidean geometry as the prime example of rigor. Will young people, then, appreciate the deficiencies in Euclidean geometry? For two thousand years the mathematicians worked with whole numbers, fractions, and irrational numbers on an intuitive basis. Despite the fact that many of them were troubled by the difficulties in the concept of the irrational number, they did not succeed in eliminating those difficulties until the sophisticated concept of Dedekind cuts or Cantor sequences was created. If it took mathematicians so long to arrive at the logical concept of the irrational number, can we believe that young people will appreciate it at once? The introduction of symbolic logic, insofar as it is an effort to provide rigor, is likewise countered by the historical fact that Aristotelian logic not only sufficed for two thousand years but, as I pointed out in another connection, still is the logic used by practically all mathematicians.

As a matter of fact, much of this rigor does little good. Let us consider the real number system. As I just remarked, mathematicians used the various types of real numbers for thousands of years and knew all along exactly what properties these numbers must have. But, having become

somewhat rigor-conscious and conscience-stricken in recent times, they decided that they must supply a logical basis for the number system. And so they started with Peano's axioms, and by introducing couples of natural numbers, couples of signed numbers, Dedekind cuts, and the like, they managed to build up a logical basis for what they had known and used all along. Having done so, they forget all about this artificial, formal, and meaningless structure and continue to use numbers on the intuitive level just as they had done prior to the logical construction. All that the rigor accomplished was that the mathematicians salved their consciences.

The presentation of mathematics in rigorous form is ill-advised on other counts. Mathematics must be understood intuitively in physical or geometrical terms. This is the primary pedagogical objective. When this is achieved, it is proper to formulate the concepts and reasoning in as rigorous a form as young people can take. But let us always remember that the rigorous presentation is secondary in importance. As Roger Bacon said, "Argument concludes a question; but it does not make us feel certain, or acquiesce in the contemplation of truth, except the truth also be found to be so by experience." Let me cite some other wise advice in this connection. Three hundred years ago Galileo said, "Logic, it appears to me, teaches us how to test the conclusiveness of any argument or demonstration already discovered and completed; but I do not believe that it teaches us to discover correct arguments and demonstrations." And Pascal said somewhat impertinently, "Reason is the slow and tortuous method by which those who do not know the truth discover it." The proper finesse rather than logic is what is needed to do the correct thing. Of recent vintage is the observation by Herman Weyl, "Logic is the hygiene which mathematics practices to keep its ideas healthy and strong." If I may exaggerate slightly, I would put it that rigor is the gilt on the lily of real mathematics.

There is another point about the introduction of rigor which, I believe, has been lost sight of. In presenting rigorous developments of the number system and of geometry we first have to make the students aware of the difficulties and then meet them. The reason for this in the case of numbers is that the students have long understood numbers intuitively through their elementary school training. In the case of geometry, students use figures which automatically take care of the very points, such as order, which the rigorous presentation covers. Hence we have to spend a great deal of time in making a student aware of what he has accepted on an intuitive or visual basis and then prove these facts. The student is unimpressed. He knew all this before. In fact, he fails to grasp just what we are after. Are we trying to prove the obvious? Surely, he thinks, we can't be that foolish.

I might say parenthetically that the entire emphasis on rigor today is most anomalous. There never was a time when the question of what is rigorous mathematics was less clear. I cannot enter here upon the history of rigor or on the difficulties in building mathematics rigorously, but I will point out that there are serious and unresolved controversies concerning what is rigorous mathematics. We do know that many mathematicians are sufficiently skeptical of whether we shall ever settle this question to make sarcastic remarks such as "logic is the art of going wrong with confidence"; "the virtue of a logical proof is not that it compels belief but that it suggests doubts"; "a mathematical proof tells us where to concentrate our doubts." Not only have the standards of rigor varied from one age to another, but today rigor in mathematical logic is one thing; rigor in foundations is another; rigor in analysis is still another; and rigor in applied mathematics is entirely different. The whole attempt to inject rigor in mathematics has amounted to picking up jewels only to discover serpents underneath.

The isolation from reality

The modernist program is deficient in another respect. It ignores completely the primary reason for the existence of mathematics and the chief motivation for the study of mathematics, namely, the investigation of nature. Mathematics is significant and vital because it is the chief instrument for the study of the physical world. On the other hand, the pure mathematics which the modernists wish to present is pointless mathematics, a manipulation of meaningless symbols which can appeal only to an esoteric group.

Let me consider an example of how far the modernists depart from reality. The real number system and Euclidean geometry are in themselves abstractions from reality, but at least students can see their relationship to the physical world or it can readily be made evident. In undertaking to teach deductive structure, the modernists propose to use abstract and artificial postulational systems. The significant systems, such as Euclidean geometry and the real number system, are too complex. It is true that these physically significant deductive systems are complex, but even if students do manage to get some glimpse of deduction from artificial systems, what will they conclude? After studying meaningless axioms and meaningless theorems, their natural conclusion will be that the whole business is meaningless.

The traditional curriculum is already meaningless, and by heading for abstract mathematics the modernists are moving farther away from reality. To teach pure mathematics apart from physical problems is to lose the gold and keep the iron in the ore of real mathematics. The meaning of mathematics, if I may be somewhat paradoxical, is not in mathematics.

Omission of the life and spirit of mathematics

Another objection to the modernists' program that is directed toward their aim to be rigorous and abstract is that it fails to present the life and spirit of mathematics. Mathematics is a creative or inventive process, deriving ideas and suggestions from real problems, idealizing and formulating the relevant concepts, posing questions, intuitively deriving a possible conclusion, and then, and only then, proving the hunch or intuitive argument deductively. Intuition and construction are the driving forces of mathematics.

All of these values are ignored in a presentation of finished, abstract, rigorous mathematics. This version is in fact an emasculation of mathematics. It perhaps shows us how questions are answered, but not why they are put or how the answers are conceived. Arithmetic, algebra, geometry, trigonometry, and the calculus did not come about by manipulating meaningless symbols or by playing games according to rules. To have students like and understand mathematics we must help them create it for themselves. As their experiences increase and as they encounter the need for stricter formulation and checks on intuition, their appreciation of the need for proof, rigor, and finished presentations will grow.

The new position of the Commission on Mathematics

I mentioned at the beginning of this talk that a month ago Dean Meder presented the forthcoming recommendations of the Commission on Mathematics and that these were not as extreme as the views I have been attacking. But I also said that these views were equally unsatisfactory. As I understand his remarks, it appears that the first year of high school algebra will hardly be changed. The word "set" is to be introduced, but that is all. If this is all, then I do not see that anything is really being recommended insofar as this year's work is concerned. The word "set" means no more than "class" or "collection." It is a word that any teacher might think of using without expecting to

be honored for thinking of it. The word is not a magic formula that will suddenly illuminate mathematics or make the subject attractive.

Insofar as the second year's work is concerned, the major change seems to be to cut down on Euclidean geometry and to substitute co-ordinate geometry for the discarded material. To my mind, this change is not significant. In the first place, if we are talking about the college preparatory group and if the first two years are intended as preparation for later work, then the amount of Euclidean geometry that would be taught is not enough. The argument that co-ordinate geometry will be used to prove further results of Euclidean geometry is not strong. Proofs by co-ordinate geometry may be more systematic but are much clumsier. Moreover, students will be frightened by the amount of algebra they have to use. Algebra is the bane of students.

The recommendations for the third and fourth years were in part agreeable. There are parts of trigonometry, such as logarithmic solution of triangles, that can be discarded, and the introduction of statistics is, I believe, warranted.

But the major criticism I would make of even this more moderate program is that if it backs away from modern mathematics it only backs into a minor reshuffling of the traditional curriculum. In view of the fact that we have failed miserably to put mathematics across with the traditional curriculum, it seems to me that the Commission's recommendations are not going to improve the teaching of mathematics.

Principles of a new approach to the curriculum

Genuine improvement of the curriculum does call for drastic revision, though not of course by resorting to modern mathematics and not by merely reshuffling the traditional mathematics. Just what should we be doing? Let me begin by stating that we have not thus far correctly analyzed the weakness in the present curriculum. The trouble is not that we are teaching outmoded mathematics, except for one or two topics. Rather, the trouble lies in the way in which we approach the material we teach. I shall try to indicate the principles which would guide a revision of our present approach.

Have you ever taken into account how intrinsically more difficult it is to arouse interest in mathematics than in history or literature? Mathematics of any kind is abstract, and the role that it plays in human thought is not a priori clear, whereas every student knows what history is about, why he is expected to study history, and what he might gain from it.

My first principle, then, is that we must begin our teaching of mathematics by seeking to arouse interest in the subject. If this is agreed upon, then we should select material which will serve the purpose of arousing interest. But what interest can young people find in simplifying fractions, in factoring, in exponents, in the quadratic formula, and in all the other dirty, intrinsically meaningless, boring processes that we teach in first-year algebra? The fact is that we have been guided in our choice of material not by the effort to arouse interest but to teach the mathematics that will be needed in the subsequent study of the subject. Our concern, in other words, has been with preparation for the future. But with such an introduction to mathematics, few students want a future in the subject.

Some people may object that we cannot afford to waste years in arousing interest. Let us suppose, to take the worst possibility, that nothing we teach in the first two years will advance the student technically. Have we then lost two years? Not at all. Since we wish to interest more students in mathematics and we now discourage and lose most of the students who take the subject, the "preparation" we now give them for advanced work is wasted and in addition we have built up a dislike that hurts us. Actually we would not lose any time. Since students work

hard at what they like, those who go on will readily make up for lost time because they have become interested. Moreover, we would be teaching some technique while solving interesting problems. Further, if we succeed in giving students some insight into the purposes and ways of mathematics and into the uses of technique, the students will really learn what they are taught, and we shall not have to repeat the meaningless techniques in intermediate algebra and college algebra courses.

My second principle is that regardless of how we arouse interest in mathematics, even if it be through games, we must supply motivation and purpose to the mathematics we teach. This means that we must motivate each topic with a genuine problem and show that the mathematics does something to solve that problem. But let us be clear on this point. For young people, at least, the motivation for a mathematical idea or method is *not* a more advanced mathematical idea or method. The motivation for learning about functions is not to learn how to differentiate or integrate functions. In view of the actual motivation for the creation of mathematics, the motivation we can use is the solution of simple, genuine, and basic physical problems. Of course the social sciences, art, and music also provide good problems. The motivation must not be confused with the mathematics that will be used to solve the problem, but the motivation should be there.

Third, since even the great mathematicians think intuitively, we must all be sure that the intuitive meaning of each mathematical idea or procedure is made intuitively clear to the students. This intuitive side of mathematics is in fact the essence of the subject. Mathematics is primarily a series of great intuitions. The way to make the meaning of an idea clear is to present it in the intuitive setting that led to its creation or in terms of some simple modern equivalent. Physical or geometrical illustrations or interpretations will often supply this meaning. Thus $s=16t^2$ is not just a quadratic function. It is a law of falling bodies, and s and t have definite and clear physical meanings. The fact that $s=16t^2$ and not $16t$ also has an important physical significance and makes the quadratic feature impressive. Numerical examples, and especially examples wherein the process or theorem fails, help to make meanings clear.

Of course something must be done to teach the concept of proof and the language of mathematics, even though symbolism and sparse presentation of proofs often conceal the ideas. Proof should be emphasized mainly in establishing what is not obvious and even runs counter to untrained intuition. Insofar as rigor is concerned, E. H. Moore's precept is applicable, "Sufficient unto the day is the rigor thereof." And by the day I mean the student's age. Fortunately, young people (and even older ones) will accept as rigorous and acquire a feeling for proof from proofs that are really not rigorous. Is this deception? I call it pedagogy. At any rate, it is no more deception than we practice on ourselves. As our own capacity to appreciate more rigorous proofs increases, we are able to see flaws in the cruder proofs taught to us and to master sounder proofs. But again let me warn you that there are no final rigorous proofs. Not all the symbolism of modern symbolic logic, Boolean algebra, set theory, postulational methods, and topology can make mathematics rigorous, and these ideas certainly do not make mathematics more understandable or acceptable to young people.

Fourth, knowledge is a whole, and mathematics is a part of that whole. Mathematics in every age has been part of the broad cultural movement of the age. We must relate the mathematics to history, science, philosophy, social science, art, music, literature, logic, as well as to any other development which the topic in hand permits. We should try as far as possible to organize our material so that the development of the mathematics

proper is related to the development of our civilization and culture. At the very least, each *major* topic should be imbedded in the cultural context which gave rise to it and should be capped by a discussion of what the creation has done to influence the development of our civilization.

I shall not attempt in this brief talk to give a detailed outline of courses. I will say that my conception of the first two years of high school mathematics would entail a radical revision of the present curriculum. I have worked on material for high school and college courses, and I am convinced that we can make mathematics attractive and meaningful. In fact, the present high position of mathematics in our culture means that the subject does have extraordinary interest, purpose, meaning, and significance. We cannot doubt, then, that it is possible to make the subject live.

Have you read?

BENJAMIN, HAROLD R. "The Saber Tooth Tiger Returns," *Indiana Teacher*, May 1958, pp. 424, 425, and 444.

This is not an article about mathematics, but those of you who have lived through several changes in curriculum will enjoy this little satire on the period from 1958 to 2008. For example, schools abdicate to nonschool agencies, mass communications, labels, and slogans. The significant trend is the central belief of the individual. Every learner is unique. Performance is only on the basis of the individual. Every school has its own program. Everyone is educated by desire. Well, this is a pleasant fantasy—read it and see.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

BRYAN, JOHN C. "Mathematics in General Education," *School Science and Mathematics*, April 1958, pp. 249–255.

Those of us in mathematics have always known that mathematics makes a contribution to general education. Mr. Bryan brings out this and other points pertinent to our mathematics curriculum. He believes that this general education in mathematics can and should be interesting and challenging. He points out how advanced mathematics may really be a clarification process, that concepts are of much value to anyone deserving a general education, and that abstraction may be fascinating to even nonmathematics-directed students. I am certain you will agree with the idea of more thorough exploration of the field, an emphasis on encouraging more critical thinking, and a tying together of all the parts of mathematics. Mr. Bryan's illustrations clarify his points.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

BUSWELL, G. T., "A Comparison of Achievement in Arithmetic in England and Central California," *The Arithmetic Teacher*, February 1958, pp. 1–9.

Today we are all looking at our programs of mathematics. Therefore, I think you will be interested in this article, which is the report of a study made in January 1957 by the Department of Education of the University of California. The study consisted of administering a 100-item mathematics test, developed in England for the examination of 11-year-olds, to English and American groups equated as to age. The groups were random samples of regular students of the 11-year age.

You will be surprised and perhaps disappointed in some of the results. For example, the English students did better in both computation and problems, the top in England was 16 points higher than the top in California, and California had more students at the bottom. There may be reasons for all this, but you should read the article and come to your own conclusions. You will be interested in the type of questions, I am sure.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

The ancients versus the moderns—a reply

ALBERT E. MEDER, JR., *Rutgers University, New Brunswick, New Jersey.*

BOTH PROFESSOR KLINE and the Editor of THE MATHEMATICS TEACHER have very courteously and generously offered to let me read the manuscript of the foregoing article and to comment upon it from the point of view of the Commission on Mathematics. Let me begin with a sincere expression of thanks for this courtesy and the evidence it constitutes that Professor Kline desires the debate to be conducted on a high level of intellectual integrity. I share this view.

May I next take up an item that may be considered a matter of personal privilege? Professor Kline refers to an article of mine in THE MATHEMATICS TEACHER of October 1957 as one from which he derived, in part at least, his understanding of the nature of the movement to teach modern mathematics in the high school. However, there is not one word in this article in this article in advocacy of the views opposed by Professor Kline in his paper. My article was not a presentation of the nature of the movement to teach modern mathematics in the high school or an argument in its behalf; it was rather an attempt to explain the nature of modern mathematics. It stated in so many words that "many teachers have been presenting traditional content from a point of view accurately described as modern"; emphasized that "the same fact can often be discussed from the traditional point of view of algebra as a form of manipulative skill, or from the modern point of view of algebra as a study of mathematical structure." It pointed out that new subject matter had developed from the axiomatic point of view, described what some of the new subject matter was, and made the following statements concerning the teaching of secondary school mathematics:

> We are asking that teachers familiarize themselves with certain basic concepts—new in form but familiar in part in substance—and creatively utilize these simplifying and unifying concepts in their teaching once again to make the study and learning of mathematics a stimulating, interesting, and vitalized intellectual adventure.

I am confident that Professor Kline does not mean to attack that kind of teaching.

There are essentially two aspects of Professor Kline's paper that I find objectionable: first, the vague generalizations, entirely undocumented, concerning the views held by the "modernists," and second, the inferences drawn from what has not been said by the "modernists." No one could object, though he might disagree, if Professor Kline quoted specific statements made by those whose views he is attacking, gave their source, and explained why he thought they were erroneous. But he does not do this. He neither documents his generalizations nor does he even define his terms. Moreover, he draws inferences from the fact that certain statements have not been made. This is indefensible.

I shall now list some of Professor Kline's undocumented generalizations and comment upon them:

(1) "The argument most often offered by the modernists is that the mathematics we have been teaching is outmoded because it is several hundred years old." I submit that this statement is not docu-

mented because it cannot be documented; nobody would be foolish enough to make such a statement.

I have myself often stated that the mathematics we teach is seventeenth-century mathematics. But this statement has never, to the best of my knowledge and belief, been coupled with the statement that *therefore* it is outmoded. Parts of what we teach certainly are outmoded, not because they are old but because they antiquated. Professor Kline is quite correct in saying that in mathematics, knowledge does not become useless because it is old. Nevertheless, it can become antiquated in the sense that it is no longer the most appropriate material for inclusion in a secondary school curriculum.

Professor Kline says: "If the argument that what is old is definitely bad is a sound one, then certainly all of the old mathematics should be thrown out." Agreed. Since, however, it is obvious that all of the old mathematics should not be thrown out, it follows that the hypothesis that what is old is definitely bad is false.

Professor Kline ends his discussion on this point by asking that the "modernists" stop emphasizing the word "modern." He thinks this word has a propaganda aspect that he regards as bad. For myself, I would be perfectly willing to eschew the word. I suspect, however, that Professor Kline would not like our ideas any better if we described them by some phrase other than "modern mathematics."

(2) "The modernists would replace material currently taught by such topics as symbolic logic, Boolean algebra, set theory, some topics of abstract algebra such as groups and fields, topology, postulational systems, and statistics."

Professor Kline does not like the word "modern" because it is "propaganda"; I do not like his word "replace" because it is not quantified. How much material currently taught is to be replaced by the topics listed? All? A great deal? A little? It makes a great difference. This generalization is so broad as to be almost meaningless.

The fact is that it is proposed by the Commission on Mathematics to include in the secondary school program, in place of some small part of material currently taught, some small part of set theory and abstract algebra. I know of no proposal to include symbolic logic, Boolean algebra, or topology. It is proposed to include statistics, but since Professor Kline agrees that this is unobjectionable, I shall not discuss this proposal.

It may be that someone somewhere wants to do some of these things that the Commission does not propose. That is precisely the point; statements should be documented.

(3) The next generalization has to do with the importance in the body of mathematics of such topics as symbolic logic, topology, Boolean algebra, and abstract algebra. Professor Kline thinks none of these is of any real significance. He is entitled to his opinion. My contention is that his opinion is irrelevant. Since it is not proposed to teach these topics as central in the development of mathematics, it is irrelevant whether Professor Kline thinks they are central or whether he thinks they are not. Actually, I think that he has less interest in some rather fascinating parts of mathematics than he ought to have. I probably have less interest in some parts of mathematics that he thinks exciting than he would think I ought to have. The fact is that no human being can possibly learn all mathematics, however much all of us would like to do so, and therefore a selection has to be made. But it is surely a rash generalization for any one of us to say that the mathematics in which we happen to be interested is more important than the mathematics in which someone else happens to be interested.

I would like to make one other remark on this point. Our opinion of the importance or difficulty of any branch of mathematics is likely to be related to the order and the time in which we ourselves

learned it. Generally speaking, subjects learned in the graduate school or from our own reading are apt to seem more remote to us than the subjects we were taught in secondary school or college. But if one learned in school or college the subjects we learned later and vice versa (assuming the possibility of doing this, to some extent), then it is altogether likely that the relative difficulty and importance of the topics would also seem quite different to such an unconventional learner. All of us need to beware of this pitfall.

While I am most certainly not advocating teaching topology and symbolic logic in junior high school, I do say that if this were done, the student who had been so taught would almost certainly have a different psychological attitude toward these subjects than do Professor Kline and I.

Finally, Professor Kline by his own admission is not a specialist in symbolic logic, topology, Boolean algebra, or abstract algebra; why not let someone who is a specialist speak concerning the importance of these subjects?

(4) The next vague generalization is that these modern topics are practically useless in their applications. I am glad that Professor Kline said, "My evaluation here is that they are practically useless." Many others would completely disagree with him.

(5) The next vague generalization made by Professor Kline is that the abstract approach to traditional mathematics is relatively inefficient. He argues that because a person understood the concept of a field he would not know how to add two and two. He states further, "A person with an excellent knowledge of fields could not even make change in a grocery store." Well, who ever said that he could? The purpose of teaching the concept of a field is not to enable a person to make change in a grocery store. Here we have two laudable objectives, although Professor Kline apparently thinks only one of them is laudable. The first objective is to teach arithmetic. We believe in this. The second objective is to teach the structure of algebra. We believe in this, too, even though Professor Kline doesn't.

(6) The next vague generalization is summed up in the question: "How can we expect students to learn the abstract before the concrete?" The answer is we can't and we don't. On this point I completely agree with Professor Kline: it is essential to teach intuitive, concrete notions before abstractions can be taught.

Moreover, on June 8, 1957, Professor Carl B. Allendoerfer, a member of the Commission on Mathematics, addressed a letter to Professor Kline that included the following statements:

> *Abstract concepts.* No one in his right mind would teach these before the concrete material. Certainly abstraction is hard, but it is still necessary. I favor introducing it earlier than is done at present, but not before the students are ready for it. Many of the laws of algebra (the axioms of a field) are easier to understand and much more useful than the arbitrary "rules" which now appear in elementary algebra books. There is no reason why they should not be taught and used—but only after the proper intuitive foundation has been laid. . . .
>
> Finally, let it never be said that anyone on the CEEB Commission favors teaching mathematics in reverse by doing abstractions first and then coming to the concrete illustrations. We do, however, favor the gradual introduction of the abstract at a rate considerably faster than is done at present. We are sure that this is both possible and desirable.

Despite this statement by a member of the Commission, Professor Kline still insists that the "modernists" advocate teaching "the abstract before the concrete."

(7) "The second major change in viewpoint which the modernists advocate is more rigor."

This generalization is not documented because it cannot be documented. No such position has been advocated.

On this point also Professor Allendoerfer stated the position of the Commission in his letter to Professor Kline almost a year before Professor Kline delivered the address on which I am commenting. He said:

Rigor. Of course the level of rigor must suit the age and maturity of the student. But this does not mean to dispense with it entirely. I think that you have missed the point on the teaching of logic. Actually it is not lost on young people. My experience has been that they lap it up. The sole object of teaching it is to help the students understand the structure of the deductive process—not to make logicians out of them. Too many students have been through geometry and only memorized the theorems. I favor a brief session on logic to let the students see what they are trying to do. The logic, however, should be brief and should not interfere with the mathematics as such.

It would take far more space than I have a right to expect to discuss Section VII of Professor Kline's paper in detail. I will therefore merely make two remarks: first, much of what he says, properly understood, is of course correct; second, it does not give, in my opinion, a true picture of the relative importance of intuition and rigor in mathematics. But this is a matter of opinion, and a discussion of it would not be germane to this debate.

(8) "The modernist program . . . ignores completely the primary reason for the existence of mathematics and the chief motivation for the study of mathematics, namely, the investigation of nature." The "primary reason for the existence of mathematics" is a matter of opinion (though I think Professor Kline is probably right about it), but surely the "chief motivation" for the study of mathematics may differ for different individuals and be differently considered by different philosophers.

Moreover, the primary reason for the existence of mathematics may not be either the primary reason for the inclusion of the subject in the school curriculum or the primary criterion for the selection of curricular content. And it is certainly an unwarranted assumption that the chief motivation for the study of mathematics is identical for all individuals. I am sure that the desire to investigate nature is the chief motivation for some students; I am equally sure it is not the chief motivation for all. This certainly was not what led me to major in mathematics, for example.

(9) Finally, there is a vague generalization to the effect that the proposed program "fails to present the life and spirit of mathematics." Then follows one of the really good paragraphs in Professor Kline's paper.

All I can say is that of course we desire to have mathematics so taught. The quotation from my own article in THE MATHEMATICS TEACHER of last October given at the beginning of this article is sufficient comment on this unjustified generalization.

Let us turn now to Professor Kline's "arguments from silence":

(1) "The modernists also give the impression that the topics they recommend are the main ones now claiming the attention of mathematicians."

Why should anyone think that topics recommended for inclusion in the secondary school curriculum are the main topics claiming the attention of mathematicians? I submit that, on the contrary, one would be justified in forming the impression that topics recommended for inclusion in the secondary school course would most likely not be topics of principal interest to research mathematicians. It is not likely that school boys and research mathematicians would have the same principal interests, nor that topics most suitable for instruction would be also those most suitable for research.

But the truth of the matter is that it may well be considered irrelevant, in a discussion of the secondary school curriculum, to comment on the activities of research mathematicians. Because irrelevant remarks are not made, it does not follow that they were not made because they were considered unimportant.

(2) "It seems curious to me that when the modernists talk about applications, they make no mention of the application of mathematics to mechanics, sound, light, radio, electricity, atomic and nuclear theory, hydrodynamics, aerodynamics, geophysics, magnetohydrodynamics, elasticity, plasticity, chemistry, chemical

physics, physical biology, and the various branches of engineering."

There is nothing curious about this situation. If one is discussing the applications of "modern" mathematics, it is not to be expected that he will talk about the applications of classical mathematics. It is unfair to chide him because he does not discuss a topic different from the one he set out to discuss.

But there is a still better explanation: rightly or wrongly, it is assumed to be common knowledge that classical mathematics has important applications to science and engineering.

Moreover, it is false to say that these applications are never mentioned. I quote from an article in the *College Board Review*, Winter 1958, an official publication of the agency that appointed the Commission on Mathematics:

> In order to formulate a curriculum oriented to the needs of the second half of the twentieth century, the Commission had to ask what these needs were. It takes no narrow view here; it is interested in the needs of mathematics itself, of course, but it is also interested in the needs of the users of mathematics, and in the applications of mathematics. . . .
>
> Thus, when the Commission speaks of the "needs" of the second half of the twentieth century, it means the needs of mathematics, *of physical science*, of social science, of technology, *of engineering*, of business, of industry [italics added].

I must now comment very briefly on that section of Professor Kline's paper entitled "The New Position of the Commission on Mathematics." First, I want to introduce a demurrer to the use of the word "new." The Commission on Mathematics does not have an old position or a new position. The position of the Commission on Mathematics has not changed. Professor Kline's understanding of it has changed, principally because he started talking about the point of view of the Commission without finding out what it was.

However, Professor Kline's statement of what he calls the new position of the Commission on Mathematics is just as distorted as his presentation of what he would have called the old position of the Commission on Mathematics. He says, "It appears that the first year of high school algebra will hardly be changed. The word 'set' is to be introduced, but that is all." This is not an accurate statement. I do, however, agree with Professor Kline that the word "set" is not a magic formula which will suddenly illuminate mathematics or make it attractive.

Again, to describe the "major change" of the geometry program as "to cut down on Euclidean geometry and to substitute co-ordinate geometry for the discarded material" is also a distortion. Professor Kline says this change is not significant; I disagree. However, that is not what the change is. The change is an attempt to teach the true nature of deductive reasoning, and also to teach a great deal more geometry more creatively than is now traditionally accomplished, as well as to make use of the fact that in the twentieth century geometry can be taught in a less cumbersome manner than in the time of the Greeks, principally because we have developed an adequate algebra.

Apparently Professor Kline does not like algebra because it is "the bane of students." I doubt that it is, but in any event this does not seem to be an adequate reason for not introducing algebraic or co-ordinate geometry into the high school program.

It is utterly amazing to me how Professor Kline can so freely criticize something of which he does not have full knowledge. He makes a few generalized, partly accurate, partly inaccurate statements concerning the "new position" of the Commission on Mathematics, and then criticizes it on the grounds that "if it backs away from modern mathematics it only backs into a minor reshuffling of the traditional curriculum."

Those of the readers of THE MATHEMATICS TEACHER who have had the opportunity of seeing the Third Draft of the Report of the Commission on Mathematics know that the recommendations

of the Commission on Mathematics are far from being a minor reshuffling of the traditional curriculum. To be sure, we do "back away" from modern mathematics in the sense that we do not "replace" the traditional material by "symbolic logic, Boolean algebra, topology, etc.," which is what Professor Kline thought we were going to do. The truth is, however, that we never had any intention of doing what he thought we were going to do; neither are we going to propose what he now thinks we are. My advice is that Professor Kline wait until he reads the Report of the Commission before he criticizes the Commission's position.

In conclusion, I will comment on Professor Kline's "Principles of a New Approach to the Curriculum." Unfortunately these principles are also vague generalizations: we must teach mathematics by seeking to arouse interest in the subject; we must supply motivation and purpose to the mathematics we teach; we must be sure that the intuitive meaning of each mathematical idea or procedure is made intuitively clear to the students; knowledge is a whole and mathematics is a part of that whole.

Well, of course I agree with all of these generalizations. But I must say that I do not agree with the presuppositions underlying these platitudes.

First, Professor Kline assumes that "it is intrinsically more difficult to arouse interest in mathematics than in history or literature." I just don't believe it. I taught mathematics for twenty-two years before becoming a full-time administrative officer; my experience is that this assumption is not true. Certainly it is correct to say that mathematics of any kind is abstract, and the role that it plays in human thought is not a priori clear. But it is not correct to say that "every student knows what history is about, why he is expected to study history, and what he might gain from it."

Second, with respect to motivation and purpose, Professor Kline believes that such motivation is to be found in the solution of simple, genuine, and basic physical problems, and he will also admit problems from the social sciences, art, and music. It is not clear whether this is the only possible motivation, or the best possible motivation. I do not agree with either assumption, however.

Finally, Professor Kline ends on the same note of vague generalization with which he started. He says that he has a conception of the first two years of high school mathematics that would entail a radical revision of the present curriculum, but he does not tell us what it is. I submit that if he has such a revision, he is under an intellectual obligation to reveal it.

Having finished Professor Kline's paper and these comments upon it, I am left with one unanswered question. I am wondering whether in point of fact, Professor Kline really likes mathematics. I do not deny that he is a mathematician, and one of considerable competence. I do not deny that he is a teacher of mathematics, and from the views he expresses concerning mathematical pedagogy, I am inclined to believe that he is a good teacher of mathematics. But I think that he is at heart a physicist, or perhaps a "natural philosopher," not a mathematician, and that the reason he does not like the proposals for orienting the secondary school college preparatory mathematics curriculum to the diverse needs of the twentieth century by making use of some of the concepts developed in mathematics during the last hundred years or so is not that this is bad mathematics, but that it minimizes the importance of physics.

The assumption can, of course, be wrong. But it is an assumption that explains Professor Kline's apparent antipathy to all parts of mathematics not directly related to physical applications, his desire to teach mathematics by means of physical problems, his feeling that mathematics is intrinsically less interesting than other subjects, and his evident distaste for abstraction and rigor.

The national high school mathematics contest

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Mathematics contests have proved to be popular. This growth raises questions concerning whether these tests should emphasize aptitude or achievement.

THE MATHEMATICAL ASSOCIATION of America, in co-operation with the Society of Actuaries, staged the first nationwide mathematics contest for high school students this last spring. On March 27, an eighty-minute objective test, consisting of fifty questions, was administered to eighty thousand students in over 2,600 high schools from Maine to California and from Ontario to Texas. It marked an important forward step in the mathematics contest movement in this country.

Two years ago a study of contests revealed the existence of nearly sixty active mathematics contests operating in the country's high schools.[1] These were serving over four thousand schools, distributed over some thirty states. They differed considerably in pattern, purpose, and type of sponsorship, but in every case were designed to promote interest and achievement in mathematics. The study showed that local conditions differed considerably between the separate communities served by these contests, and that any attempt to standardize a contest pattern on a nationwide basis would be fraught with serious difficulties, if not to say insurmountable obstacles. In fact, a list of ten arguments was presented against the case for a nationwide contest program.

The same study, however, revealed that there was a receptive attitude toward contests in general; that in over a fourth of the states no program existed at all; that there was a need for further development of contests, particularly in this virgin territory; and that a nationally organized program could possibly serve a need, if designed cautiously, and with deference to already existing programs.

The implications and results of this study were weighed thoughtfully at the 1956 Seattle meeting of the Mathematical Association. As a result of this analysis, and upon the favorable report and recommendations of their Committee on High School Contests, the Association voted to sponsor a nationwide contest in 1958.

The past experience of the Association in the field of high school contests had centered principally in the program operated annually by the Metropolitan New York Section since 1950. The New York Committee,[2] headed by Professor William H. Fagerstrom and Charles T. Salkind of New York, had operated its own local

[1] Daniel B. Lloyd, "The National Status of Mathematics Contests," THE MATHEMATICS TEACHER, XLIX (October 1956), 458–463.

[2] The 1956 members of this committee were: Brother Bernard Alfred, William Allan, William H. Fagerstrom, Howard F. Fehr, K. G. Fuller, Kenneth B. Morgan, Barnett Rich, Myron F. Rosskopf, Charles T. Salkind, Ramon Steinen, and Ellsworth E. Strock.

contest there, from the first in 1950 to its eighth in 1957, and had also provided the examination questions for similar contests in five other states and one province of Canada. In all, it had reached a maximum of forty thousand students by 1957. Also the Maryland-District of Columbia-Virginia section had successfully conducted a similar contest annually in their three states since 1954.[3]

The plans for the first nationwide contest were started in the summer of 1957 by a Committee on High School Contests appointed by MAA President G. Baley Price of the University of Kansas. He appointed the following members: H. L. Alder, University of California, Davis, California; William Allan, Actuary, Home Life Insurance Co., New York, New York; A. J. Coleman, University of Toronto, Ontario, Canada; William H. Fagerstrom,* Pan American College, Edinburg, Texas; Mildred Keiffer*, Board of Education, Cincinnati, Ohio; Daniel B. Lloyd,* D. C. Teachers College, Washington, D. C.; D. C. Murdoch, University of British Columbia, Vancouver, Canada; Charles T. Salkind,* Polytechnic Institute of Brooklyn, New York; L. F. Scholl,* Board of Education, Buffalo, New York; Sister Mary Felice,* Mt. Mary College, Milwaukee, Wisconsin; C. F. Stephens, Morgan State College, Baltimore, Maryland; E. E. Strock, Actuary, Prudential Life Insurance Company, Newark, New Jersey; Arnold Wendt, Western Illinois University, Macomb, Illinois.

Among this committee of thirteen, it will be noted that two members are from the Society of Actuaries; and six are members of both NCTM and MAA. The Society of Actuaries is underwriting the initial expense of the contest, although the program is expected to be self-supporting as soon as it is well established.

The following principles were laid down for the contest program for the Association's twenty-seven regional sections:

1. Each Section is to decide whether it will sponsor the national contest within its territory—either independently or jointly with other organizations.
2. If the Section votes to sponsor the contest, a local Standing Committee on High School Contests should be appointed by the Chairman of the Section. This Committee together with the Committees of other sponsoring organizations, if any, form the "Sectional Contest Committee."
3. If a Section votes against the sponsoring of a national contest, it may initiate or continue a local contest in mathematics for high school students (either independently or with other organizations), and the national contest should not be conducted within the territory of the Section. The same principle should be followed in cases where a Section decides not to sponsor the national contest because other organizations are conducting a local contest in that area.
4. Under any other circumstances, the National Committee may set up its own procedure for taking care of the requests of individual schools of that area, with the proviso that it shall not compete or interfere with existing local contests conducted by the MAA or other organizations therein.

Before the start of the 1958 contest, thirteen Sections had voted to enter it, eight Sections were studying it further, three wished to continue the contest programs they already had for another year, and three had taken no action. Thus, the reaction nationally seemed quite favorable. At any rate, the National Contest Committee found themselves very busy.

As actually carried out, the contest was conducted in nineteen separate areas—some of the MAA Sections being subdivided geographically for contest purposes by their own contest committees. For in-

[3] Daniel B. Lloyd, "A New Mathematical Association Contest," The Mathematics Teacher, XLVIII (November 1955), 469–472.

* Also members of The National Council of Teachers of Mathematics.

stance, the Pacific Northwest Section is divided into four units, Idaho, Oregon, Washington, and British Columbia; and the upper New York Section is divided into a Buffalo region and an Albany region.

Professor Fagerstrom, Chairman of the Committee, had retired from his teaching position in New York and accepted a position as Professor of Mathematics at the Pan American College in Edinburg, Texas. There he set up the contest headquarters and directed the work. One subcommittee on test construction designed the examination. Another subcommittee on publicity distributed brochures to the schools. The mammoth task of organizing and conducting the first nationwide contest was thus completed on schedule.

The tests are given at, and by, the schools themselves, on the morning of the scheduled day. The results are scored by the school and the three highest papers are sent to the Contest Committee. The total of these three scores constitutes the school score, or the team score.

Some of the 1958 test questions are on page 439. The test is designed like most standard tests with multiple-choice responses to facilitate scoring. The questions cover high school algebra and geometry—but not trigonometry or advanced algebra, as such. To discriminate between the keenest students, the test is considerably longer than the average student has time to complete. It becomes a power and speed test for the majority. This should be understood by the schools and not cause discouragement. The teachers administering the tests should carefully explain to the competing students that they are not expected to complete the test, but merely to answer as many questions as time permits.

The test score is computed as follows: $S=R-\frac{1}{4}W$. In this formula, R represents the number of points allowed on the problems correctly answered; W represents the total points for the problems incorrectly answered. This formula, statistically speaking, largely eliminates any advantage in guessing answers. The answer sheet is detachable from the test booklet. This is submitted by the competing student; the test booklet may be retained by the student for review purposes for the following year.

Students in any grade of high school may compete. Although an advantage is enjoyed by those with more mathematical training, winners occasionally emerge from the lower semester students. This is because the tests are designed to measure innate ability as well as training. Also, the wide variety of questions helps to equalize the opportunity for schools with different curricula.

The number of tests ordered by each school varies considerably. This indicates a great diversity of philosophy among the schools as to the use of the tests. In some schools there is apparently a thorough elimination and selection on the local level, only the very superior students being allowed to compete. In others, the test is widely administered, and possibly used for pupil evaluation or other purposes, as well as for the contest. In any event, the incentive of achievement prevails, and the basic purpose of the contest is attained.

Each school in the National Contest pays a $4 registration fee, which covers the cost of the award to the school winner and entitles the school to twenty examination booklets. Additional copies, if desired, may be bought at ten cents each. The school designates it own winner and returns the three highest papers to the National Committee. The winner is awarded a pin with a safety catch (or a lapel button, if preferred) bearing a facsimile of the seal of the Mathematical Association of America. In case two students are tied for winner, each may receive the same award, if the school wishes to pay $1.75 for the extra award.

For the purpose of awards, the United States and Canada were divided into nine regions, as follows:

REGION	
I	New England States and Canadian Maritime Provinces
II	Upper New York State and Eastern Canada
III	Metropolitan New York and New Jersey
IV	Middle Atlantic States
V	The South
VI	Middle West States and Central Canada
VII	Mountain and Southwestern States
VIII	Northwestern States and Provinces
IX	Pacific Coast

A certificate of merit is awarded to the schools in each region scoring in the highest ten per cent of team totals. In addition to this, a bronze cup is awarded to the highest-ranking school in each region. The Chemical Rubber Company donated one hundred copies of mathematical tables, which were awarded equitably to the highest students in each region.

Caution is exercised not to embarrass smaller schools or those less capable in competition with larger ones. For instance, no over-all lists of scores are published, and no national awards are made. In fact, every school has a winner, and receives at least one award.

Some of the general results of the 1958 contest were as follows (maximum possible individual score is 150):

Highest individual score	146.25[4]
Lowest individual score	2.00
Highest team score (3 members)	371.75
Lowest team score	2.00

The distribution nationwide (only the three highest in each school are submitted) showed these medians and quartiles:

	Q_3	Md	Q_1
Individual scores	42	31	24
Team scores	106	79	60

Only 77 students made a score of 90 or above, which placed them on the Honor Roll.

Considerable flexibility characterizes the administration of the contest program in the various sections of the association. Some sections prefer to handle their own program, making use only of the national committee's examination booklets. In this case, the section purchases the booklets at $60 per thousand, with no other obligations except to administer the examination professionally and on the same scheduled date as the national contest. They may vary the financial charges to suit their own needs. The local section collects the winning papers and makes its own awards. The awards in the form of pins or buttons, each bearing a facsimile of the seal of the Mathematical Association of America, may be purchased from the L. G. Balfour Company, Jewelers.[5] Some of the sections have built up an extensive system of local awards, financed by local interested parties such as business houses, corporations, banks, or other organizations, as well as scholarships in certain colleges and universities.

The National Council of Teachers of Mathematics seriously considered assuming the responsibility of launching a national mathematics contest itself. In 1950, its Board of Directors officially approved ". . . a national mathematics contest for high school seniors, to be conducted with the co-operation of the MAA." The purpose set forth was ". . . the improvement of instruction in mathematics, encouragement of students of real promise to continue the study of mathematics at the college level, and emphasis on the importance of mathematics in modern scientific and industrial life."[6] No further action was taken on this plan, although it was never actually rescinded.

The study on contests made in 1956 recommended the policy and program which has since been adopted by both the NCTM and MAA.[7] It pointed to the advantages of the MAA's assuming an independent responsibility for such a contest, citing the following reasons:

[4] This highest score was made by a student in Illinois.

[5] L. G. Balfour Co., Attention: Mr. Schmeelk, 521 Fifth Ave., New York 17, N. Y.

[6] NCTM Board Minutes, April 12, 1950, Chicago, Illinois.

[7] See Footnote 1.

(1) The MAA membership is more detached from the competing schools and therefore can handle the contest more impartially.
(2) Joint, or divided, administration would be more cumbersome and less efficient.
(3) Persons with valuable contest experience, who might be most helpful, are members of both organizations and already are active in contest work.

The study further predicted, "It seems possible that such a project may get started in a couple of years. It could well emerge as an outgrowth of the present sectional contests, particularly the Metropolitan New York one, now the largest in the country. It could have great potential in attracting support from wealthy foundations or corporations."

The predicted support has been made a reality through the generosity of the Society of Actuaries. It is likely that more support will be forthcoming. With the heightened emphasis on mathematics and science, federal as well as private sources have been considering numerous ways to stimulate the interest of young people in these fields. A recent bill introduced in the lower house of Congress suggested a payment of $500 to every high school student passing a standard examination in mathematics. Though not yet enacted into law, it received some strong support and clearly reflects the thinking of the times.

The National Mathematics Contest program is designed to enhance interest and improve achievement in high school mathematics. In the awards, the philosophy has been to provide prestige and satisfaction to worthy students, encouraging *all* competitors as much as possible, but avoiding large monetary awards.

When this year's results are studied and the opinions of the participating schools are analyzed, certain improvements can be made for next year's test. Some schools have stated that the test is too long for the time allowed; others say it is too difficult. Thus, many competitors may be discouraged. The answer that the committee gives to this is that a few of the competitors scored near the maximum, so that an easier test would not separate them. These are matters that the committee must weigh carefully in designing next year's test. Large numbers of students should not be unduly thwarted for the sake of a few.

The experience gained in the 1958 contest has prompted the Committee to make some slight changes in the 1959 edition: (1) the examination itself will be a bit easier and shorter to encourage more schools to enter; (2) awards will be made to the schools earlier in the semester; (3) the contest will be given in *two* stages, for the sake of some sections desiring both. The first stage, on March 5, will be similar to the contest in 1958. The second stage (the finals) for the upper 5 or 10 per cent of the winners will be given at testing centers in specified colleges under monitored conditions on April 15. The second stage of the contest will serve as the basis for certain final awards and scholarships. It will be run entirely by the local sections desiring it—not by the National Contest Committee. This double-stage contest pattern was proposed by Prof. R. C. Buck, of the University of Wisconsin.

Ultimately there must evolve an improved contest program that will best serve the national and local interests. Like the satellite program, it cannot be expected to emerge perfectly in its first launching effort. It is a thing new and big and must be gradually improved experimentally. Constructive criticism from all sides should be both expected and respected. Now adequately financed and in professionally competent hands, a national program in some form is here to stay, at least for some years to come.

The NCTM has endorsed the program both officially and unofficially. Upon the inauguration of the program, Dr. Howard Fehr, then president of NCTM, said, "Members of the Council are urged to take the necessary steps to secure participation of their schools in this contest." Our current president, Professor Harold P. Fawcett, Ohio State University, has also endorsed it in similar terms.

Further information on the 1959 contest can be obtained from Professor W. H. Fagerstrom, at the Pan-American College, Edinburg, Texas.

Below are sample questions from the 1958 examination:[5]

4. In the expression

$$\frac{x+1}{x-1}$$

each x is replaced by

$$\frac{x+1}{x-1}.$$

The resulting expression, evaluated for $x=\frac{1}{2}$, equals: (A) 3 (B) -3 (C) 1 (D) -1 (E) none of these

13. The sum of two numbers is 10; their product is 20. The sum of their reciprocals is:
(A) 1/10 (B) $\frac{1}{2}$ (C) 1 (D) 2 (E) 4

18. The area of a circle is doubled when its radius r is increased by n. Then r equals:
(A) $n(\sqrt{2}+1)$ (B) $n(\sqrt{2}-1)$
(C) n (D) $n(2-\sqrt{2})$ (E) $\frac{n\pi}{\sqrt{2}+1}$

43. AB is the hypotenuse of a right triangle ABC. Median $AD=7$ and median $BE=4$. The length of AB is:
(A) 10 (B) $5\sqrt{3}$ (C) $5\sqrt{2}$
(D) $2\sqrt{13}$ (E) $2\sqrt{15}$

[5] A package containing the 1958 examination, together with a solution booklet, may be obtained for 25¢ from the Mathematics Contest Committee, Polytechnic Institute of Brooklyn, Brooklyn 1, N. Y. Sample copies of old contest examinations from 1953 to date are available in limited numbers at 25¢ for the first three copies and 6¢ for each additional copy.

What's new?

BOOKS

Secondary

Arithmetic in My World, grade 7, C. Newton Stokes, Paul J. Whiteley, and Humphrey C. Jackson. Boston: Allyn and Bacon, Inc., 1958. Cloth, 384 pp., $3.04.

Arithmetic in My World, grade 8, C. Newton Stokes, Paul J. Whiteley, and Anne Beattie. Boston: Allyn and Bacon, Inc., 1958. Cloth, 383 pp., $3.04.

Basic General Mathematics, Margaret Joseph, Mildred Keiffer, and John Mayor. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1958. Cloth, iv+458 pp., $3.52.

Business Mathematics (5th ed.), R. Robert Rosenberg and Harry Lewis. New York: Gregg Publishing Division, McGraw-Hill Book Company, Inc., 1958. Cloth, xvi+560 pp., $3.84.

A Classbook of Arithmetic and Trigonometry, S. F. Trustram and H. Whittlestone. London: G. Bell and Sons Ltd., 1958. Cloth, xxxii+368 pp.+lxxxv, 12*s*. 6*d*.

A New Geometry for Secondary Schools (3rd ed.), Theodore Herberg and Joseph B. Orleans. Boston: D. C. Heath and Company, 1958. Cloth, ix +422 pp., $3.20.

College

Algebra: A Textbook of Determinants, Matrices, and Algebraic Forms (2nd ed.), W. L. Ferrar New York: Oxford University Press, 1957 Cloth, vii+220 pp., $2.80.

Analytic Geometry, Edwin J. Purcell. New York: Appleton-Century-Crofts, Inc., 1958. Cloth, x+289 pp., $4.50.

Analytic Geometry Problems, C. O. Oakley. New York: Barnes and Noble, 1958. Paper, xviii+253 pp., $1.95.

Application of Tensor Analysis (Dover republication), A. J. McConnell. New York: Dover Publications, Inc., 1957. Paper, xii+318 pp., $1.85.

Arithmetic for Colleges (rev. ed.), Harold D. Larsen. New York: The Macmillan Company, 1958. Cloth, xiii+286 pp., $5.50.

Basic Mathematics, H. S. Kaltenborn, Samuel A. Anderson, and Helen H. Kaltenborn. New York: The Ronald Press Company, 1958. Cloth, ix+392 pp., $4.75.

Basic Mathematics for College Students, Paul A. White and Raymond C. Perry. Dubuque, Iowa: Wm. C. Brown Company, 1957. Paper, viii+189 pp., $2.50.

Calculus of Variations and Its Applications (proceedings of symposia), Lawrence M. Graves, editor. New York: McGraw-Hill Book Company, Inc., 1958. Cloth, v+153 pp., $7.50.

College Algebra (4th ed.), Joseph B. Rosenbach, Edwin A. Whitman, Bruce E. Meserve, and Philip M. Whitman. Boston: Ginn and Company, 1958. Cloth, xiv+579 pp. +xlvi, $5.25.

Polynomial expansions and a polynomial distribution theorem

LLOYD SEDGWICK *and* KAREN MCQUISTON, *Eastern High School, Greentown, Indiana.*

Here is an interesting use of the binomial theorem and its application to certain problems in probability.

AN INTERESTING ADAPTATION of the binomial theorem to a trinomial expansion consists of a series of binominal expansions such as:

$$(a+b+c)^n = (a+b)^n + n(a+b)^{n-1}c + \cdots + c^n$$

Expanding the binomial factors of each term gives us:

$$\begin{aligned}(a+b+c)^n &= a^n + na^{n-1}b + \cdots + b^n\\ &+ na^{n-1}c + n(n-1)a^{n-2}bc + \cdots + nb^{n-1}c\\ &+ \cdots\\ &\quad \cdots\\ &\quad \cdots\\ &+ nac^{n-1} + nbc^{n-1}\\ &+ c^n\end{aligned}$$

It will be noticed that not only is each series a binomial expansion with the exponent of a in the first term in each series one less than in the preceding series, but the first terms themselves form a binomial expansion containing $n+1$ terms, with a and c as the factors. Since each row contains one fewer term than the preceding row, starting with $n+1$ terms in the first set and diminishing to one term in the final series, then we see that the total number of terms in a trinomial to the nth power is $\frac{1}{2}(n+1)(n+2)$.

Similarly, $(a+b+c+d)^n$ can be shown to be a series of $n+1$ trinomial expansions, such as:

$$\begin{aligned}(a+b+c+d)^n &= a^n + na^{n-1}b + \cdots + b^n\\ &+ na^{n-1}c + n(n-1)a^{n-2}bc + \cdots + nb^{n-1}c\\ &+ \cdots\\ &\quad \cdots\\ &\quad \cdots\\ &+ c^n\\ &+ na^{n-1}d + n(n-1)a^{n-2}bd + \cdots + nb^{n-1}d\\ &+ n(n-1)a^{n-2}cd + n(n-1)(n-2)a^{n-3}bcd\\ &+ \cdots + n(n-1)b^{n-2}cd\\ &+ \cdots\\ &+ \cdots\\ &+ \cdots\\ &+ d^n\end{aligned}$$

Perhaps a better way to represent this expansion would be a triangular pyramid, with each level a trinomial expansion, decreasing to the one term in the vertex (d^n). Again it will be noticed that the first terms of each trinomial expansion themselves form a binomial expansion of $n+1$ terms with a and d as the factors. Thus a 4-nomial to the nth power would contain $n+1$ trinomial expansions. If the top level of the pyramid is the expansion of $(a+b+c)^n$, then all the other levels combined would have the same number of terms as $(a+b+c+d)^{n-1}$, and it can be seen that the number of terms in the expansion of a 4-nomial to the nth power is the sum of the number of terms in the expansion of a 4-nomial to the $(n-1)$ power

and the number of terms in a trinomial expanded to the nth power.

It would take a 4-dimensional pyramid, a figure we've been calling a "tetrad," to represent a 5-nomial expansion, a "pentad" to represent a 6-nomial expansion, and the series would continue in this way.

The number of terms in a 4-nomial expanded to the nth power is

$$\frac{(n+1)(n+2)(n+3)}{2\cdot 3}.$$

By the associative law, any polynomial of m terms could be expanded in the same manner. But rephrasing the binomial theorem to state that the coefficient of any term may be derived by multiplying the coefficient of the preceding term (or the first term of the preceding series) by the exponent of a in that preceding term, and dividing by one more than the exponent of the factor which is increasing in that series, might make the polynomial theorem easier to apply.

Since a polynomial of m terms expanded to the nth power would be a series of expansions of a polynomial of $m-1$ terms with the total number of terms in the m-nomial expansion the sum of the number of terms in each of the $(m-1)$-nomial expansions, then the numbers of terms in an m-nomial expanded to the nth power is the sum of the number of terms in an m-nomial expanded to the $(n-1)$th power and the number of terms in an $(m-1)$-nomial expanded to the nth power.

This means that we can form a table taking the form of a Pascal triangle standing on one of the base angles, where m equals the number of terms in the polynomial and n equals the power to which the polynomial is expanded, and the numbers in the table are the number of terms in each expansion. (See Table 1.)

An interesting formula which will determine the number of terms in the expansion of a polynomial of m terms to the nth power is

$$T=\frac{(n+1)(n+2)\cdots(n+m-1)}{(m-1)!}.$$

M =	1	2	3	4	5
N = 0	1	1	1	1	1
N = 1	1	2	3	4	5
N = 2	1	3	6	10	15
N = 3	1	4	10	20	35
N = 4	1	5	15	35	70
N = 5	1	6	21	56	126

Table 1

This formula could be used in probability problems as an extension of the binomial distribution. Thus, if an event could occur in six different ways (such as rolling a die), then for seven successive events, where we are concerned only with combinations and not permutations, the number of different combinations would be

$$\frac{(8)(9)(10)(11)(12)}{(2)(3)(4)(5)},$$

or 792. The probability of any one such combination can be determined by the exponents and coefficients of such term in a 6-nomial expansion.

That is, if a, b, c, and d each have a probability of .2 of appearing on any single throw, and e and f each have a probability of .1 of appearing on any one throw, the probability of getting an a, three c's, and three e's in seven throws could be computed by finding the coefficient of the ac^3e^3 term in a 6-nomial expanded to the seventh power. This term could be found either by combinations or by the method suggested earlier in this article, and would be 140 ac^3e^3. Next we substitute .2 for a and for c, and .1 for e in this expression to determine the probability of this combination. That is: $P=140\,(.2)(.2)^3(.1)^3$ or $P=.000224$. It is interesting to note that the number of terms in the expansion is the number of combinations of such events, while the sum of the coefficients is the number of permutations.

Pascal's triangle of binomial coefficients can also be adapted to a pyramid illustrat-

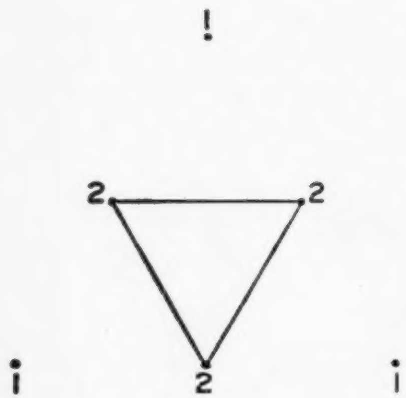


Figure 1

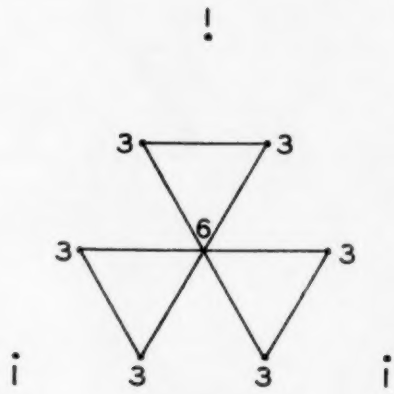


Figure 2

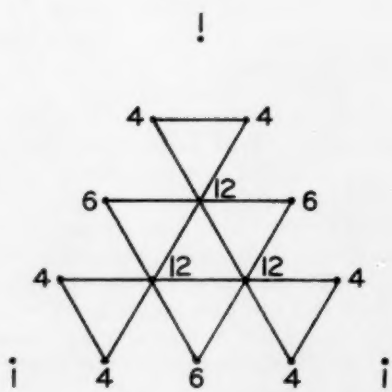
ing the coefficients of a trinomial expansion. This pyramid is a triangular pyramid with each lateral face a Pascal triangle and the coefficients of each trinomial expansion indicated by a set of equilateral triangles at regular intervals within the pyramid. The terms inside the triangle of each level are derived by adding the coefficients of three adjacent terms from the preceding level.

That is, the vertex of the pyramid would represent $(a+b+c)^0$. The first level would be a triangle with a 1 at each vertex, representing the coefficients of $(a+b+c)^1$. The second level would be six points arranged as in Figure 1, representing $(a+b+c)^2$, and the third and fourth levels would be as in Figures 2 and 3, respectively.

The lines drawn in each level indicate the triangle whose vertices determine the coefficients in the following level. The nearer a point is to the center of the triangle, the larger its coefficient, and points equidistant from the center represent equal coefficients. Similarly, the nearer a point is to one vertex of the triangle forming that level, the larger the exponent of that factor in the term, and conversely.

Similarly, other "pyramids" could be built using n-dimensional geometry. That is, if $m=4$, we would have a "tetrad" with each level a tetrahedron, and if $m=5$, we would have a "pentad" with each level a "tetrad", and continuing in the same manner.

Figure 3



The limit

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"So you've got to the end of our race-course?" said the Tortoise. "Even though it does *consist of an infinite series of distances? I thought some wiseacre or other had proved that the thing couldn't be done?"*—Lewis Carroll.

THERE IS A DEFINITE TREND in secondary school mathematics education toward introducing the elements of the differential and integral calculus in the twelfth year. The teacher who is some years removed from his undergraduate calculus studies is confronted with the problem of refreshing his knowledge of calculus in order to deal competently with the problems of syllabus construction, lesson planning, and actual classroom instruction involving calculus. The objective of this article is to review one of the basic analytic ideas underlying the differential and integral calculus. This is the concept of the limit.

The concept of the limit is central in calculus, since both the derivative and the definite integral are really limits. Every time anyone calculates a derivative or a definite integral, he is finding a limit.

Let us imagine a conversation between a student and his calculus teacher who is trying to clarify some of the student's hazy notions about limits.

STUDENT: The text talks a lot about limits. What is a limit really?

TEACHER: Your use of the word "really" indicates that the *mathematical* concept of limit is somehow inadequately described by the common word "limit" from our English language. This is undoubtedly true, as the technical use of the word implies an understanding of mathematical concepts not at all evident from the word itself. Let me give you a few examples of limits first, and then we'll try to formulate the general concept. Here is a sequence of numbers: $1/1$, $1/2$, $1/3$, $1/4$, $1/5$, $\cdots$. The general term of the sequence is $1/n$ where n takes on successive positive integral values. What would you say is the limit of the sequence? I don't mean the sum of the terms but rather the limit of the individual term as we go farther out in the sequence.

STUDENT: I'd say the limit was zero, since the successive terms are getting smaller and smaller as n increases.

TEACHER: That's right. The limit is zero, but the mathematician sees much more in the situation than just that. Does any specific term ever *equal* zero?

STUDENT: No, because even if n is, say, one billion, the number $1/1{,}000{,}000{,}000$ is not zero, although very close to it.

TEACHER: How close?

STUDENT: Well, if the terms were points on a number scale with units of, say, centimeters, the point for $n = 1{,}000{,}000{,}000$ would be one billionth of a centimeter from zero.

TEACHER: And what about all the terms in the sequence whose n is greater than one billion?

STUDENT: They would be even closer to zero.

TEACHER: Now we have been talking about the limit in a mathematical manner. Suppose I were to name a small number, say, 0.001. What value would n have to be

so that the difference between $1/n$ and the limit zero would be less than 0.001?

STUDENT: The value of n would have to be greater than 1000, that is, 1001, 1002, etc. Then 1/1001, 1/1002, and each succeeding term would be closer to zero than 0.001.

TEACHER: In other words, no matter how small a number is chosen, isn't it always possible to find a place in the sequence beyond which all the terms of the sequence are closer to zero than the chosen small number?

STUDENT: Yes. As I begin to see it, the mathematician is not just interested in the limit itself, which in our example is surely zero, but in the closeness to it of something else, in this case $1/n$.

TEACHER: Exactly. The mathematician says that zero is the limit of $1/n$ because $1/n$ can be made as close to zero as anyone desires simply by choosing n large enough. Now note this fine, but very crucial, point: $1/n$ never actually equals zero no matter how large n is.

STUDENT: Do you mean that a variable can have a limit and never actually be equal to the limit?

TEACHER: That is precisely the idea. The actual attainment of the limit value by the variable is not necessary at all.

STUDENT: It seems to me that you go to a lot of trouble to have the variable close to its limit, and then do not care much at all about whether it ever really equals the limit.

TEACHER: Well, that's the whole point. In our example, $1/n$ can't ever equal zero, but it can be as close to zero as we want. As you will see in later developments, particularly when we study the derivative and definite integral, this aspect of the limit concept is very fruitful. What is probably disturbing you is a seeming nonmathematical lack of preciseness in the idea of the limit. Up to now the equality sign has played a dominant role in your mathematical thinking. You are used to saying, for example, $x=0$, or $x=4$, which sounds somehow solid and reassuring. Now you are meeting a more subtle type of mathematical thought. Something does not *equal* something else but merely is very *close* to it. These qualms, which you seem to have, are to be expected. We get comfortably used to an idea, or way of thinking, and then suddenly we are confronted with a situation that requires a new approach. Naturally, you are somewhat resistant to the disturbance of your set thinking pattern. But these are only intellectual growing pains.

STUDENT: Could we look at another specific example of a limit?

TEACHER: Of course. The example I have in mind now arises from a most interesting paradox from the Greeks which indicates that even those formidable thinkers wrestled with the difficulties of the limit concept. Have you ever heard of Zeno's paradox of Achilles and the tortoise?

STUDENT: I've heard of Achilles as a great hero in the Trojan War.

TEACHER: Well, Zeno said that Achilles, great as he was, could not catch the slow tortoise if the tortoise had an initial head start, because if both Achilles and the tortoise started running at the same time, by the time Achilles reached the point where the tortoise started, the tortoise would have crawled to a new position. Then by the time Achilles reached that position, old tortoise would be at still another position. In other words, a faster body can never overtake a slower body because by the time it reaches any position of the slower body the slower body will have moved on.

STUDENT: But that's ridiculous.

TEACHER: Of course it's ridiculous, and the reason for its being so is that we are playing fast and loose with words when we should be treating the situation by mathematical analysis. I think you will admit, though, that old Zeno jolts us a bit by showing how dangerous it is to "verbalize" a problem. Now let's analyze his paradox by making a specific problem of it. Instead of Achilles and the tortoise,

we'll consider body F (for fast) and body S (for slow). Suppose that at the time $t=0$, S is at a position one mile ahead of F. Then the distance between them is $d=1$. Now let S move (in a straight line) at the rate of one mile per hour, while F moves at two miles per hour. How long will it take F to get to S's original position, and where will S be then?

STUDENT: It will take F $\frac{1}{2}$ hour, since F's speed is 2 miles per hour and he has to travel 1 mile. By that time S will have gone $\frac{1}{2}$ mile, and the distance between them will therefore be $\frac{1}{2}$ mile.

TEACHER: That's right. Now keep your eye on the distance between them. At the start $t=0$, $d=1$; then at $t=\frac{1}{2}$, $d=\frac{1}{2}$. Now repeat the process. F will take only $\frac{1}{4}$ hour to reach S's second position and by that time S will have moved on another $\frac{1}{4}$ mile. Thus at $t=\frac{3}{4}$, $d=\frac{1}{4}$. A similar calculation will show that at $t=\frac{7}{8}$, $d=\frac{1}{8}$, and so on. In other words, the sequence of successive distances between the moving bodies is

$$1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \cdots \text{ or}$$

$$\frac{1}{2^0},\ \frac{1}{2^1},\ \frac{1}{2^2},\ \frac{1}{2^3},\ \cdots \frac{1}{2^{n-1}}\ \cdots,$$

recognizing that the general term is $1/2^{n-1}$. For example, in the fourth term where $n=4$, we have $1/2^3$.

STUDENT: This is like the first example, because the limit of the nth term is surely zero, since as n gets larger $1/2^{n-1}$ gets smaller. Or, as you would want me to say it, for any chosen number, no matter how small, a certain term in the series can be found such that the difference between it and zero is smaller than the chosen small number.

TEACHER: Right, but you should add that the difference between all *succeeding* terms in the sequence and zero is also smaller than the chosen small number.

STUDENT: But even if the limit of the successive distances between the two bodies is zero, we have not shown that F really overtakes S. In other words, no matter how big n is, there is always a little d left.

TEACHER: But we have shown that the "littleness" of d can be controlled. That is, it can be made smaller than the smallest number you can think of, simply by taking n large enough.

STUDENT: But according to the line of argument, d never does equal zero.

TEACHER: The distance d does not equal zero so long as the time t is less than 1 hour, since at time $t=1-(1/2^{n-1})$, n terms in the d sequence have been generated and $d=1/2^{n-1}$. But time inexorably marches on. When t *equals* 1 hour, as it inevitably will, n becomes infinite, and then we associate the limit value of zero with the distance d. Here, then, is a situation where the mathematical limit has a very real physical significance since in the finite time of 1 hour an infinite number of terms of the d series are generated. Therefore it is the limit of d which we have to consider, and the limit does *equal* zero. Thus Achilles does catch the tortoise in spite of Zeno's objections.

STUDENT: Are there any other examples of limits?

TEACHER: There are many kinds of limits. So far we have discussed only sequences of terms. Let's look now at a limit of a sum of terms, also known as the limit of an infinite series. We can get this out of the Achilles-tortoise type problem which we have had under consideration by asking how far F moves from his position at $t=0$ before he overtakes S.

STUDENT: Well, we should get this by adding up the successive distances that F moves, in other words, the successive d's. Now we want the limit of the *sum* of the d's rather than the limit of the individual terms. That should be $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots 1/2^{n-1}+\cdots$ and so on indefinitely. But that's easy. I recognize that as a geometric series with a ratio, $r=\frac{1}{2}$, and first term, $a=1$. Therefore, the sum is $s=a/(1-r)$, $=1/(1-\frac{1}{2})$, $=2$. The answer is 2.

TEACHER: If by "answer" you mean the result of inserting numbers into the for-

mula $S=a/(1-r)$ and doing a little arithmetic, then I must admit you have "solved" the problem. But we are not now primarily interested in answers, but rather fundamental concepts. Can you express your ideas in terms of limits?

STUDENT: Let's see. The difference between the sum of the first n terms in the series and 2 can be made smaller than any positive number you can name.

TEACHER: That's fine. You surely have the spirit of the limit concept. Let me try to put it into more precise mathematical language. Let us define $S_n \equiv$ sum of the first n terms of the series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots 1/2^{n-1}+\cdots$. Thus, for example,

$$S_4=1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}, \text{ and}$$

$$S_5=1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}, \text{ and}$$

$$S_n=1+\tfrac{1}{2}+\tfrac{1}{4}+\cdots 1/2^{n-1}.$$

Notice that the series for S_n terminates at $1/2^{n-1}$. Now let me try to translate your previous statement into analytic symbolism. You said in part, "The difference between the sum of the terms of the series and 2," and this I shall write as

$$|S_n-2|.$$

The absolute value signs are used because it is the magnitude of the difference, regardless of its sign, that interests us. You next said that this difference "can be made smaller than any positive number you can name." The mathematician's favorite symbol for this small number is ϵ, the small Greek epsilon. Thus you say

$$|S_n-2|<\epsilon$$

where ϵ is an arbitrarily small, positive number. I'd like to carry this a bit further by saying that for any ϵ you choose, I can find a specific value of n which will make this inequality true. For example, if you choose $\epsilon=0.1$, then I note that

$$|S_4-2|=|1\ 7/8-2|=.125,$$

and

$$|S_5-2|=|1\ 15/16-2|=.0625.$$

Now $|S_6-2|$ and every succeeding $|S_n-2|$ will surely be smaller than .0625. Therefore, for all values of $n \geq 5$, the inequality

$$|S_n-2|<0.1$$

is true.

STUDENT: We are right back to the idea of controlled closeness as in the previous examples.

TEACHER: Right. Now let me try to say this all at once: If

$$S_n=1+\tfrac{1}{2}+1/2^2+1/2^3+\cdots 1/2^{n-1},$$

then for any arbitrarily small positive ϵ, a specific value of n, say N, can be found such that whenever $n \geq N$, then

$$|S_n-2|<\epsilon.$$

STUDENT: The specific value of N depends on the choice of ϵ?

TEACHER: That's right. In fact you can say that N is a function of ϵ or $N=f(\epsilon)$. The actual solution of a limit problem is not just to find the limit, which may be fairly easy, as it was in this case, but rather to find this functional relation between the small number ϵ and some other parameter which controls the range of the variable of problem. In the case we have considered, this other parameter was N, for only when the variable n was greater than or equal to N, was $|S_n-2|<\epsilon$. Therefore the real problem here is to find N as a function of ϵ so that when a specific ϵ is chosen, an N is determined such that when $n \geq N$, then $|S_n-2|<\epsilon$.

STUDENT: That sounds as if it might be a real tough problem.

TEACHER: In general it is, and for the present at least I would like to defer it until we develop the concept of the limit of a function with a continuous independent variable. Let me then put this problem to you: Find

$$\lim_{x\to 2} x^2,$$

where x can have any real value.

STUDENT: I see that x is a continuous variable which makes this different from

the discrete cases we have already considered. I am pretty sure that the answer to the problem is 4, but I know by this time that what you want me to discuss is the closeness of x^2 to 4 when x is close to 2. I feel that it would be helpful to tabulate some values.

TEACHER: Good idea. May I suggest the following tabular arrangement?

(1) x	(2) x^2	(3) $\lvert x-2\rvert$	(4) $\lvert x^2-4\rvert$

STUDENT: I can see the necessity for the first two columns, but why do we want columns (3) and (4)?

TEACHER: Well, column (3) will measure the closeness of x to 2 and column (4) will measure the closeness of x^2 to 4.

STUDENT: I see. Now I'll substitute some values for x close to 2 and see what happens.

(1) x	(2) x^2	(3) $\lvert x-2\rvert$	(4) $\lvert x^2-4\rvert$
2.2	4.84	.2	.84
2.1	4.41	.1	.41
2.01	4.0401	.01	.0401
2.001	4.004001	.001	.004001

TEACHER: That's enough to show some significant trends.

STUDENT: Yes, now I see why you put in columns (3) and (4). When x is within .001 of 2, x^2 is within .004001 of 4. In other words, if you name the number .004001 as the desired closeness of x^2 to 4, I can guarantee that closeness simply by taking x within .001 of 2.

TEACHER: You are almost correct and I hate to mention this, but what if $x = 1.999$?

STUDENT: Oh yes! We must consider that, of course. My calculations show that when $x = 1.999$, $x^2 = 3.996001$; $|x-2| = .001$ and $|x^2-4| = .003999$, so I am still right! If you make .004001 the closeness criterion for the function, then .001 is the closeness criterion for x regardless of what side of 2 x lies on.

TEACHER: Can you generalize these thoughts? Suppose instead of a specific small number .004001, I merely gave you the general symbol ϵ, implying an arbitrary small positive number. What would you have to find?

STUDENT: I would have to find another small positive number, say, δ, such that whenever x is within δ of 2 the function x^2 would be within ϵ of 4.

TEACHER: In other words, for any arbitrary small positive ϵ, a δ can be found such that when $0 < |x-2| < \delta$, then $|x^2-4| < \epsilon$.

STUDENT: Is that what the mathematician has in mind when he writes

$$\lim_{x \to 2} x^2 = 4?$$

TEACHER: Exactly.

STUDENT: One thing still puzzles me. If the problem "Find

$$\lim_{x \to 2} x^2\text{"}$$

were put to you, you wouldn't actually go through the process of constructing the tables or investigating the ϵ,δ relation; you would just substitute 2 for x in x^2 and get 4, wouldn't you?

TEACHER: Yes.

STUDENT: Isn't that being a little intellectually dishonest to find the limit by letting x equal 2 when the whole philosophy of the concept of the limit is that x can't equal 2?

TEACHER: That is a sharp question which shows that you are beginning to develop a healthy critical state of mind. I'll try to answer you by showing that there are some situations where the limit can be found by direct substitution quite legitimately, and other situations where it is impossible. As a counterexample consider this problem: Find

$$\lim_{x\to 0} \frac{\sin x}{x}.$$

STUDENT: When $x=0$ is substituted in $\sin x/x$ we get $0/0$, which is meaningless. Yet I recall that this limit equals 1.

TEACHER: Yes, and it is found by a geometric proof. How about this one,

$$\lim_{x\to 2} \frac{x^2-4}{x-2}?$$

STUDENT: Again indeterminate.

TEACHER: Now try

$$\lim_{x\to 2} \frac{4}{x-2}$$

by direct substitution.

STUDENT: This would be $4/0$, which is meaningless.

TEACHER: Can you see how each of these counter examples differs from

$$\lim_{x\to 2} x^2,$$

which can be evaluated by direct substitution?

STUDENT: Well, something strange happens to the function at the limit; it either is indeterminate or infinite.

TEACHER: The mathematician would say that the function does not then *exist*. Under those circumstances the limit can't be found by direct substitution.

STUDENT: Then so long as the function exists at $x=a$,

$$\lim_{x\to a} f(x)$$

can be found by direct substitution of $x=a$ in $f(x)$? In other words,

$$\lim_{x\to a} f(x)=f(a)?$$

TEACHER: The existence of the function is only a *necessary* condition. There are still other conditions which have to be met before you can safely find the limit by substitution. *All* of these conditions together establish the continuity of the function at $x=a$. If the function is continuous at $x=a$, then

$$\lim_{x\to a} f(x)$$

does equal $f(a)$, and one can use substitution of $x=a$ to find the limit.

STUDENT: What are these conditions for continuity?

TEACHER: You can determine if a function $f(x)$ is continuous at $x=a$ by determining (1) if $f(x)$ is defined at $x=a$, that is, if $f(a)$ exists; and (2) if

$$\lim_{x\to a} f(x)$$

exists. If $f(a)$ exists and

$$\lim_{x\to a} f(x)$$

exists, and if

$$\lim_{x\to a} f(x)=f(a),$$

then you have shown that $f(x)$ is continuous at $x=a$. If it happens that $f(x)$ is not defined at $x=a$, or that

$$\lim_{x\to a} f(x)$$

does not exist, or that, in case both exist, they are different numbers, then you have determined that $f(x)$ is discontinuous at $x=a$. I realize that this is pretty technical, and perhaps some day we should discuss the concept of continuity in greater detail. For the present we can say very roughly that a function is continuous at a point with abscissa $x=a$, if its graph has no breaks there. However, the really rigorous definition of continuity proceeds as I first indicated.

STUDENT: There is one more aspect of limits that I would appreciate your showing me. Would you work out the functional relationship between ϵ and δ for a specific problem?

TEACHER: Surely. The very heart of any limit problem is to know what δ one needs in $|x-a|<\delta$ to meet the requirements of any ϵ in $|f(x)-L|<\epsilon$. I'm using L as the symbol for the limit of the function $f(x)$,

as $x \to a$. For a specific problem, consider the old stand-by:

$$\lim_{x\to 2} x^2 = 4.$$

Our problem now is not to find the limit 4. It is easy to see that x^2 is close to 4 when x is close to 2. What we want to examine is the relation between $|x^2-4| < \epsilon$ and $|x-2| < \delta$. In other words, to find δ as a function of ϵ, so that when ϵ is specified as the desired closeness of x^2 to 4, δ will be determined as the necessary closeness of x to 2.

STUDENT: In other words, if someone gives you some value for ϵ like .001, then the function of ϵ could be used to find the numerical value of δ, which would be the range around 2 within which x would have to lie to make x^2 lie within .001 of 4.

TEACHER: That puts it very well. Now see if you can follow the analysis where I use the general symbol ϵ instead of .001.

$$|x^2-4| < \epsilon$$

can be written

$$-\epsilon < x^2 - 4 < \epsilon,$$

since x^2-4 can be either positive or negative but not larger in magnitude than ϵ. This inequality is satisfied by the same values of x if 4 is added throughout. Therefore, we have

$$4-\epsilon < x^2 < 4+\epsilon.$$

Taking the positive square root throughout, we get

$$\sqrt{4-\epsilon} < x < \sqrt{4+\epsilon}.$$

This inequality expresses the range of positive x consistent with

$$|x^2-4| < \epsilon.^*$$

Now, surely,

$$\sqrt{4-\epsilon} < 2 < \sqrt{4+\epsilon},$$

since ϵ is positive. All we have to do now is to see which of $\sqrt{4-\epsilon}$ and $\sqrt{4+\epsilon}$ is closer to 2. Then this range will equal δ. A little thought will tell you that

$$|2-\sqrt{4-\epsilon}| > |2-\sqrt{4+\epsilon}|,$$

since $\sqrt{4-\epsilon} < \sqrt{4+\epsilon}$. Therefore, $\sqrt{4+\epsilon}$ is closer to 2, and we finally get $\delta = \sqrt{4+\epsilon}-2$ as the desired functional relation.

STUDENT: That was rough.

TEACHER: Yes, unfortunately each limit problem presents its own unique difficulties in finding the ϵ, δ relation.

STUDENT: I may not be able to reproduce that development myself, but I'm pretty sure I know its meaning.

TEACHER: What is that?

STUDENT: Well, if ϵ is given as, say, .01, we can now calculate δ exactly as $\delta = \sqrt{4.01}-2 = .00250$, and we can then say that if $|x-2| < .00250$, then

$$|x^2-4| < .01.$$

TEACHER: That's it. We now have the complete picture about

$$\lim_{x\to 2} x^2 = 4.$$

We not only know that the limit is 4, but with the relation $\delta = \sqrt{4+\epsilon}-2$ we also know exactly what δ to use for a given ϵ to be able to say that when $|x-2| < \delta$, then $|x^2-4| < \epsilon$.

STUDENT: What is there to learn next about limits?

TEACHER: Well, the derivative and the definite integral are both limits. If you'll drop in again sometime, we can see how the limit concept is applied to those two basic ideas of calculus.

* Actually, this range only holds if $0 < \epsilon \le 4$. When $\epsilon > 4$, we have $-\sqrt{4+\epsilon} < x < \sqrt{4+\epsilon}$. This, however, leads to the same ϵ, δ relation as the inequality considered.

Grouping
–in the normal mathematics class

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Meridian Public Schools, Meridian, Mississippi.

An account of how one high school surveyed its mathematics program and how it used the results of its survey.

As a part of the curriculum restudy initiated in the Meridian Public Schools in 1955, an experimental revision of the mathematics program, grades one to fourteen, is under way. Experimental revision in the mathematics area is to be centered around three areas:

1. A study of mathematics requirements in general education.
2. An examination of the newer content suggested for mathematics today.
3. A cautious examination of the methods for meeting individual differences—homogeneous and heterogeneous grouping, acceleration and/or enrichment, and individualizing instruction within a class.

A survey of the mathematics program in the Meridian Public Schools was made. Data obtained were:

1. There was evidence that the achievement levels of the various grades from one through fourteen were at or above the national norm on standardized tests. The range of scores was very narrow, being no more than two grades and usually one grade between the highest and the lowest student score.
2. The Iowa Algebra Aptitude Test was given a selected group of thirty top students from the sixth grade of the nine elementary schools in the city. These scores were compared with the scores of students on a similar test given at the end of the eighth grade. The thirty students in the sixth grade scored as high as the eighth-grade students, with the exception of the very top scores.
3. There was indication from a survey made that the elementary school teachers and high school teachers used the normal everyday occurrences in arithmetic, or mathematics, for extending or enriching the abilities of students in mathematics.
4. It seemed that most teachers followed a single textbook as a guide, with all students in each grade or in each section progressing generally at the same rate.

Experimentation procedures

In the school year 1956–1957, two teachers at the senior high level began experimentation with grouping in first-year algebra, plane geometry, second-year algebra, and business mathematics. No attempt was made to introduce new content into these subject courses. Instead, the two teachers made work sheets involving all the chapters in each book. Each student, or group of students as grouping developed, was allowed to work at his own rate of speed in moving through the prescribed material in the course. The teachers moved from group to group giving such instruction or assistance as was needed. No attempt was made to use large or whole group activities throughout the year.

Results of experimentation

The following results were obtained. No more than six students in each group moved beyond the prescribed course work. These six students went as far as approximately one-third year in the succeeding courses. For example, the first-year algebra students completed first-year algebra and went into plane geometry. The students in plane geometry moved into second-year algebra. Second-year algebra students and business arithmetic students did not progress beyond the accepted course-matter material but did complete all of the sections in the book. The books in the second-year algebra and business arithmetic have sections which are called enrichment or supplementary materials.

Conclusions and recommendations

At the end of the school year, the two teachers analyzed the year's work in order to plan for 1957–58. Their analysis was as follows:

1. Three sections of students after a trial period of grouping stated that they did not like working in groups and worked throughout the remainder of the year as a whole group. Three other sections of students also stated that they did not like to work in small groups and started out as a whole group. However, a small group of six to twelve students in each section got bored with the whole process and decided to work on by themselves. They proceeded at their own rate of speed, while the majority of students continued together. Four sections of students worked completely in small groups with three to five groups operating independently at the same time.

   It was extremely difficult to get the very superior child to move far out ahead of the other students. The stigma of "brains" or "egghead" or "bright boy" was too much for most students. Homogeneous grouping does not seem to be the answer, as such taunts are thrown at members of a homogeneous section of bright students in the English field.
2. Students in sections where grouping occurred proceeded at varying rates of speed. At certain times they would speed up and work as far ahead as two or three chapters. Then they would coast for a time. In the end, the majority of students completed the usual class work even though many small groups of the brighter students completed all of the text material. Only six students moved into the next year's course, and those people worked as individuals.
3. There was a more relaxed atmosphere in the classroom. The students helped one another often. They felt that they could work as they wished and did not feel the compulsion of the teacher-compelled requirements.
4. There was a small group of normal students who failed because they could not assume the responsibility. Whether or not they could have passed under normal circumstances is not known.
5. A large amount of materials is needed to satisfy the fast-moving students. Teachers bought several workbooks, texts, and other material and spent many extra hours preparing work sheets.
6. The students stated in their evaluation at the end of the year that they would have liked some activities which pulled them together as a whole group. They felt they kept "their nose to the grindstone" too much. The teachers agreed with the evaluation of the students.
7. The teachers felt that the students worked with more speed and understanding of their own volition throughout the course.
8. Student reaction as a whole was completely favorable.
9. Parent and community approval was general.

There are two other major problems of acceleration within the normal classroom. The first problem is that most faculty

members will have to undertake this process in order to meet the individual differences encouraged in achievement; however, it must be noted that children are not going to accelerate themselves to any great degree. The second problem is that of providing whole-group activities and/or enrichment activities for the entire class. Teachers and students seemed to feel a great need for the common experience. Study has been underway this summer to devise whole-group activities which may be used in advanced mathematics classes. The study is now in process and its results will be reported on at a later date.

Have you read?

DILLE, JOHN. "The Missile-Era Race to Chart the Earth," *Life*, May 12, 1958, pp. 124–138.

What is the exact size and shape of the earth? This question is asked not only by your mathematics students but by modern scientists, too. John Dille has written an interesting and informative article on how charting the earth is being done. Did you know the unmapped portion of the world is greater than the mapped? Did you know that no single meridian has ever been measured from pole to pole? Did you realize that to get the circumference of the earth to the nearest one-tenth mile we need to correct for bending of the sun's rays? You will be amazed to read that the geodesists have found the best way to map the earth is to ignore it. Your students will appreciate the problems of traveling all over the globe to make measurements. This article describes frontiers still open and how mathematics has a basic part in their exploration. Read it and let your imagination do the rest.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

HOPPER, MARYLYN, "Geometry Plain and Simple," *Sunday Magazine, Indianapolis Star*, February 23, 1958, pp. 12–15.

This is a magazine writer's report on the activities of Madge Masten, teacher of mathematics in Plainfield, Indiana. For over thirty years this teacher has been inspiring students to do more and more in mathematics. Her students want to study mathematics. She relates mathematics to the students' world through regular projects ranging all the way from curve stitching to completed scale-model houses used by architects. Her classroom is a fascinating exhibit of mathematics in many different forms. Pupils spend their spare time browsing in her room. A local lumber dealer offers prizes to students with the best scale houses. All of this is outside the regular class activities of developing the fundamental principles of mathematics. The proof of her classroom teaching lies in the number of her students who are successful in mathematics and sciences as their life's profession. This article has implications for all of us.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

MUNROE, M. E., "Bringing Calculus Up-to-Date," *The American Mathematical Monthly*, February 1958, pp. 81–90.

It is impossible to give you the real impact of this article in a short statement. However, this is one of the few articles to bring the calculus out of cold storage and specifically show the way toward its modernization. For example, the author points out the need to distinguish between a function f and its values $f(a)$, and to note that co-ordinate variables x and y are mappings and that the differential geometry meaning of $y=f(x)$ is more adaptable in modern calculus.

The meaning of functions, the ideas of differentials, and even the simple notations need to be studied so that the meaning becomes clear and more precise than has been the case in the past. Improvement in calculus will not come by injecting more function theory, but it will come through better use of differential geometry. This will be difficult, but failure to do so means transmitting to the next generation only that information available to our grandfathers. For those of us who studied calculus ten or more years ago this article is welcome.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

Micky's proof of the medians theorem

M. L. KEEDY, *University of Maryland, College Park, Maryland.*
This true story makes a plea for the adoption of teaching techniques involving informal deduction to avoid stifling creativity in young minds.

MICKY IS FOURTEEN. He usually fails, or nearly fails, arithmetic. He tends to be a dreamy sort—forgets to do assignments and just can't see the point of learning multiplication tables. His spelling is atrocious, and although he is fairly certain that 6×8 is 48, he is quite uncertain about 8×6.

Micky, along with another boy his own age, was shown a sketch of a triangle in which the medians were drawn. It was remarked that they appeared to be concurrent, but it was pointed out that no drawing is completely precise, and the question was posed as to whether the lines were actually concurrent. The language used was much less impressive than this, of course.

After a few minutes of thinking, and a little argument between the boys, Micky announced his conclusion: The medians of any triangle were concurrent. When asked how he knew this, he replied that in any triangle one can always join the feet of the medians to form another triangle within the first, and that the medians will be inside it. (He did not realize that the medians of the two triangles actually coincide.) Then one can join the feet of the medians of the small triangle to form a third triangle within the second, and the process can be repeated. Then he announced, "Since there is no end to this, the medians have to meet in a point."

Although Micky's proof would hardly be considered rigorous by the mathematical analyst, it can be made rigorous with a bit of doing. The idea can also be used with a geometric progression to prove that the intersection of the medians trisects them. To Micky, however, his proof was more convincing than any of the traditional proofs could have been. Any chain of reasoning which was not at least partly his own would have left him in the same frame of mind as the usual arguments in favor of learning the multiplication tables. Furthermore, his reasoning, informal though it was, was principally *deductive*, and it clearly indicates the presence of an ability to reason mathematically.

It is no secret that this came as a great surprise. In fact, the reasoning involved appears unusually sophisticated, even for much more astute youngsters of fourteen. It might be added that Micky is currently struggling with mathematics homework almost nightly, attempting to finish the required number of problems and write them down in exactly the right form, to avoid a failing mark for the semester. The conclusion is obvious, perhaps even well-worn: a staggering emphasis on rote learning is stifling creativity in the young mind. In this respect Micky is not an isolated case. He may not even be in a minority.

Micky's proof of the medians theorem resulted from his being given the opportunity to do some *informal deduction*—that is, informal reasoning without reference to axioms and with no requirement that statements and reasons be written in a rigidly formal fashion. Yet the reasoning is clearly deductive, as the presence of the words "for *any* triangle" will testify.

Reasoning of this sort is sound mathematical thinking, and junior high school students might well be encouraged to do it. The teacher who encourages this kind of thinking is sure to learn some quite unconventional mathematics. It seems sure, too, that he will be providing a sound base upon which formal mathematical deduction may later be placed.

It is not easy for any teacher consistently to plan activities so as to capitalize on the creativity of children. There is no formula for such planning, nor can there be, since the creative mind, by definition, will give rise to the unexpected. It is not difficult to find hints on the use of the "discovery approach" or articles on the distinction between inductive and deductive thinking. The difference between the concrete and the abstract in teaching is still discussed at some length, and is universally known. The place of inductive thinking in school mathematics is likewise well known, but the concept of *informal deduction* is not. Micky's proof prompts a discussion of it.

There is wide agreement that in the teaching of mathematics those procedures which are sound from the mathematical viewpoint tend to be the best also from the psychological view. If this is the case, then it appears that learning should proceed in the following steps:

1. Induction
2. Deduction
3. Possible application.

New mathematics is developed in this order. Mathematicians think in this way. Application is listed only as "possible," since not all mathematics finds application. Furthermore, application is not essential to the process of learning a mathematical concept. The deduction may be formal or informal, depending on the maturity level of the student. For very young children, it may be necessary to omit this step, but it seems clear that deduction should take its proper and central place at the earliest possible maturity level. Micky's case provides a convincing argument that this level is found in early junior high school or below.

The teacher using this procedure in the junior high school would very probably introduce a new concept by using a questioning technique, getting students to express opposing views on the new question. He would encourage hunches, trial-and-error procedures, and even experiments by the students, in an attempt to answer questions. This is Step 1, Induction. When a question is formulated and there is a conviction among students that they have found its answer inductively, it is then the task of the teacher to shake this conviction, to prepare the way for informal deduction. If students have measured angles of triangles, or torn off their corners and matched them with a straight angle, to convince themselves that the sum of the angles is 180°, for example, the teacher must—unpleasant as it may seem—point out that there are "millions and millions" of triangles, whereas only a few have been tested. Furthermore, these procedures are only approximate, so the question is not really answered. It is difficult to shake an adolescent conviction, but shaken it must be before the informal deduction takes place.

When the way has been cleared for informal deduction, the teacher points out that the question must be reasoned out. True, it may be fairly certain what the answer is, but to be *sure*, reasons must be given which apply to *all* cases in question, rather than to just a few. Reasons which can be given are diverse, even for a single proposition. In the case just mentioned, a student may wish to imagine himself walking around a triangle, knowing that when he walks completely around *any* triangle, he will have turned through 360°. Or he may wish to consider a line through one vertex parallel to the opposite side, or a parallelogram made up of two triangles. The teacher need not—indeed, *must* not—have all the ideas. The students will supply them. Micky's proof emphasizes this.

The teacher must, however, insist that whatever is presented be an argument which begins "*For any* . . . ."

If applications of the propositions thus proved are to be made, habits formed in doing informal deduction should put the student in an advantageous position. He will expect to have to figure out how to make the application, instead of asking for an artificial rule. Indeed, he already has a kind of rule in the proposition that he has deduced, and it really means something to him because he has figured it out, or helped to figure it out.

It should be noted that the procedure outlined by these three steps is opposed to the classical view that learning should proceed through the concrete to the abstract. Induction may involve the concrete world, but this is not necessary. For example, a student may come to believe inductively that the sum of two odd numbers is even, without using piles of apples or any reference to the real world. Deduction is of course abstract, and the applications may be made to the real world or within the realm of mathematics itself. Thus the concrete may enter the process in Steps 1 or 3, or may be totally absent.

These three steps may be considered a modification of the so-called "discovery approach." Certainly the student is encouraged to make discoveries in all three steps. However, the usual view of the discovery approach has it based almost entirely on induction. Deduction is reserved for the more austere mathematical regions, such as formal geometry. In these steps, the idea of discovery is extended to include deduction, formal or informal, depending on the maturity of students, and even to include the area of applications.

Letter to the editor

Dear Sir:

As a member of the Thirty-sixth Convention of the National Council and one who spent considerable time trying to analyze the mood of the assembled delegates from all over the nation, I should like to report here a few findings, and to suggest a course of action.

Mathematics is presently in the national limelight, and if we are not careful, we are quite likely to take steps merely for the sake of demonstrating some kind of activity without giving proper thought to the outcome. What we need is not "new" mathematics, except in the sense that we need a "new" sunset, or "new" symphonic interpretations occasionally. The really new that is needed in mathematics is the extension of the traditional, not because it *is* the traditional, but because the traditional *is* mathematics.

We are presently faced with a just demand for acceleration in our math programs. As thinking people, we should now be giving all our thought to the development of the advanced-standing scholars in our institutions, offering them an opportunity to move rapidly through the fields of basic and advanced ideas to the natural limits of their ability. We should avoid notions and schemes which slow such scholars down, or which must be discarded when more advanced work is encountered.

These people are the elite of our school populations. We should treat them as carefully as we would handle a cherished treasure. There may be places where we should make changes in our methods of teaching them, but the changes *must be reduction* of unnecessary complications in the teaching process rather than the addition of devices or tricks not universally applicable.

I call upon the National Council to examine with a full and unhampered conscience its position with regard to the "modern" techniques of teaching ninth-grade algebra. If there is no position, I call upon you to take one, as the acknowledged leader in the profession. This is not the time for us to rally, starry-eyed, around the first camp to propose something new. Intelligent leadership here can avoid many mistakes, and of all times, now is when we should avoid them.

Yours very truly,
Arthur E. Cebelius
Ass't Principal
Sedgwick Junior High School
West Hartford 7, Connecticut

Let's try the "fi" system

DAVID H. KNOWLES, *Samuel Ayer High School, Milpitas, California.*

In these days it is good to have suggestions that bring out fundamental ideas for the schools. Teachers should do more with enumeration systems.

IN THE ACCOUNT THAT FOLLOWS no attempt will be made to justify or prove the effectiveness of the techniques shown. I shall let my experiment stand on its merits as an intellectual gambit with the properties of number systems, from which, if you are so inclined, you may construct your own tests and draw your own conclusions.

I suggest that in terms of class participation and understanding the procedure that follows can be equally successful with any group from capable fourth-graders to university graduate students. The degree of interest and participation, strangely enough, seems to depend less upon intellectual capacity than upon personality traits. Some earnest students of lower ability will grasp the ideas quickly, while more capable but practical-minded students may become impatient and perhaps consider you a little "wacky."

In using the "fi" system with general mathematics and algebra classes, covering a wide range of ability, it was found that *acceptance* of this experimental approach is not so much a matter of ability as it is a matter of attitude. Naturally *success* must hinge first upon acceptance, so we find that individual progress does not always follow expectations based upon ability. Of course, this is also true to a less notable degree in the normal course of class activity.

The purpose of the "fi" system is to develop an appreciation of number systems and some of their simpler properties. Little equipment is needed, but three dozen children's alphabet blocks or the equivalent are helpful. A proper introduction would be some discussion of primitive methods of counting, such as marks, notches, tallying, use of fingers or toes, reference to the primitive "one, two, many," and a discussion of how the problems of extending these ideas needed to be solved as counting needs grew more complex and communication extended beyond the direct person-to-person stage.

After the need has been established, then the idea "Let's invent a number system of our own" can be introduced. Can such a system be a logical development of these primitive forms? To what might we naturally relate and extend our number system? To the hand? To both hands? Might one hand be simpler? What do we need for long-distance communication? Symbolic representation? If so, what sort? Individual symbols? Words? Let's start from the primitive hand and objects to be counted and devise symbolic words for our system (Fig. 1).

We have exhausted the possibilities of one hand. We have larger quantities to represent. Where do we go from here? A whole hand was "fi". Let's start over again. A whole hand and one more would be "fi un", then "fi du", and so on to two hands. How many are two hands? Refer-

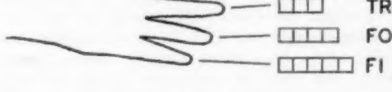


Figure 1

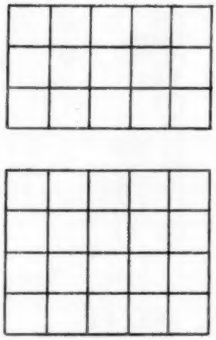
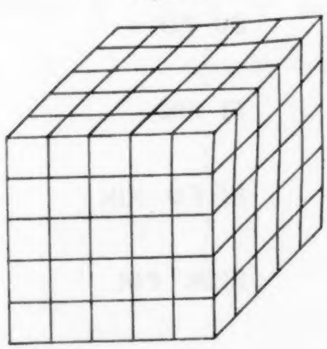


Figure 2

ring to our words and objects, we see that "du fi" accurately represents the quantity we have in mind. Our counting can continue apace to establish "tre fi" and "fo fi" (Fig. 2).

It will be simpler if, as we establish these numbers, squares are drawn on the blackboard or blocks are used. When five "fi's" are reached the answer to the question, "What sort of an arrangement do we have here?" is obvious. We may then ask, "Will 'fi square' be equivalent to a square 'fi' on a side?" "Can we contract 'fi square' to a more convenient usage?" "How about 'fisk'?" So we establish the name for a group of twenty-five as well as for a group of five. We continue our counting, illustrating with blocks from "fisk un" to "fisk fi", and on to "du fisk", "tre fisk", "fo fisk", and until we have the array (Fig. 3).

Figure 3



Whereupon, "What does it look like?" and other leading questions soon have "fi cube" contracted to "fik". By now light is dawning, and the phonetic fireworks of "fi fo", "fo fi", "fik fi", "du fik fi du", and "fi fo fik" temper and stimulate our not-*too*-serious intellectual exercise as they make their appearance.

But wait a minute! What about "none"? Must we be able to represent this? Is it a number? Let's use "oh" for void, emptiness, complete lack.

But is it not clumsy to have to write numbers in this longhand form at all times? Can our system be further simplified? How? By symbolic numerals? How did we, as primitives, indicate how many dinosaurs we saw in the glen? Did we indicate them with sticks in the dust and further tally marks (Fig. 4)?

I, II, III

Figure 4

Or did we indicate them as fingers and other hand symbols (Fig. 5)?

Figure 5

Perhaps our invented symbols should approximate these. Let's begin. (See Figure 6.)

Note that it is worthwhile to list the "agreements" we must reach from time to time to make our number system effective.

While some may object to the representation of *objects* in Figure 6, if they are introduced and recognized as a cruder symbolism or hieroglyphic, they should be acceptable to the most rigorous mind—and clearly an aid to student understanding.

OBJECTS	SYMBOL	WORD
NONE	0	OH
□	I	UN
□□	N	DU
□□□	M	TRE
□□□□	8	FO
□□□□□ = ○	I0	FI
○□	II	FI UN
○□□	IN	FI DU
○□□□	IM	FI TRE
○□□□□	I8	FI FO
○○	N0	DU FI
⋮	⋮	⋮
○○○	M0	TRE FI
⋮	⋮	⋮
○○○○	80	FO FI
⋮	⋮	⋮
○○○○○ = △	I00	FISK
△□	I0I	FISK UN
⋮	⋮	⋮
△○	II0	FISK FI
⋮	⋮	⋮
△○○○□□	IMN	FISK TRE FI DU
⋮	⋮	⋮
△△△○□□□□	MI8	TRE FISK FI FO
⋮	⋮	⋮
△△△△△ = ⊡	I000	FIK
⊡□	I00I	FIK UN
⋮	⋮	⋮
⊡○	I0I0	FIK FI
⋮	⋮	⋮
⊡⊡	N000	DU FIK
⋮	⋮	⋮
⊡⊡⊡⊡⊡ = ⊙	I0000	FI FIK
⋮	⋮	⋮
⊙⊡⊡⊡⊡	I8000	FI FO FIK
⋮	⋮	⋮
⊙⊙⊙⊙⊙ = △	I00000	FISK FIK
△△△△△ = ⊡	I000000	

Figure 6

O 1 N M & 1O 11 1N 1M 1& NO N1 NN NM N& MO M1 MN MM

Figure 7

By now the pattern is established, and the segment of your group that is not amused and confused will be amused and eagerly with you. Depending on the group with whom you are working, this may be enough exploring for one or more days. If you are assignment-minded (or would like to try for voluntary efforts), you might provide these guiding problems:

1. Can you construct a number scale for the first 150 number symbols as in Figure 7 above?
2. Make a logical construction of names for 8, 9, 10 place numbers.
3. Can you write the proper word symbols for the 150 number symbols of Question 1?
4. Change the number symbols in Figure 8 to our decimal counting system.

11; 1O1; &OON; M&N; NO&M

Figure 8

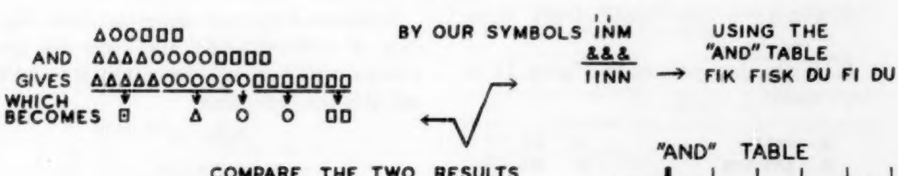
5. Change 5; 11; 21; 36; 104 to the "fi" counting system.
6. At this point, make a comparison and criticism of the "fi" and decimal number systems.

If you feel like carrying on a second day —and I hope you will—you can begin by producing the number scale of Problem 1 and having class members recall various names, after reviewing group sizes if necessary.

Now the idea of "anding" can be introduced, that is, when we have one group of objects *and* another, how can we express their *togetherness*? By the use of the number scale, it is easy to count each group, and it won't be long before someone suggests counting one group on the scale, then continuing from the end point and counting the second group. At this point, commutative and associative laws of addition can be verified and agreed upon.

But isn't it too much work to have to refer to a number scale for each "and" operation? What about finding the result of &O1MN and 11M&O? Would it be feasible to count this out on a scale?

Figure 9



"AND" TABLE

AND	O	1	N	M	&
O	O	1	N	M	&
1	1	N	M	&	1O
N	N	M	&	1O	11
M	M	&	1O	11	1N
&	&	1O	11	1N	1M

An addition table can now be developed, with emphasis on the idea of combining *like* groups. The use of different-sized blocks or symbolic notation as shown for "objects" above will aid and illustrate the "replacement" (carrying) of five smaller groups by one of the next size. (See Figure 9.)

Now consider the process of "removal," such as "having fi, remove fo," and the answer is readily found to be "un" by several methods, e.g., directional counting, reference to the addition table, and addition to the number being removed. All of these methods will emphasize and reinforce desired ideas about number systems.

What if more is to be "removed" than is present? After consideration of this problem, we can agree upon an extension of our number system and a type of notation that will satisfy our requirements. Suppose the symbols in Figure 10 are the

$$\bar{\mathrm{I}}, \bar{\mathrm{N}}, \bar{\mathrm{M}}, \bar{\&}$$

Figure 10

symbols for this extension. We can explore this as far as we like, perhaps calling our original system "right" numbers and the new system "left" numbers.

Now we may be ready for another assignment:

7. Complete an "and" table from "oh" to "fi".
8. Write the symbols in Figure 11 in words.

A. IO&MN
B. &NMOIIN
C. MOI
D. &&
E. I&&

Figure 11

9. Perform the following "ands" (Fig. 12):

Figure 12

A. IO AND &
B. NI AND IM
C. I&M AND MOI
D. M& AND IM
E. II& AND &NO AND M&&

10. Devise a symbol for "and".
11. Devise a symbol for "remove".
12. Perform these removals (Fig. 13):

A. FROM IO REMOVE &.
B. " I& " IN.
C. " &MI " MIO.
D. " III " &&.
E. " MOI " NM&.
F. FROM & REMOVE IO.
G. " IN " I&.
H. " MIO " &MI.
I. " && " III.
J. " NM& " MOI.

Figure 13

13. Make a comparison of "and" problems in the "fi" system to those of addition in the decimal system. Discuss the advantages one system may have over the other.
14. Consider the problems of "anding" and "removal" of either left or right numbers in any order. Can you devise some satisfactory rules for this that we can agree upon?

According to the amount of time you wish to spend and the apparent value your students may be receiving, you can continue on to other ideas.

Is it possible that often we shall have to find the result of the repeated "anding" of a specific group size, such as "fo" groups of "tre"? Is there any way of doing this without counting "tre" "fo" times along the number scale? Or referring "fo" times to the "and" table?

Some student may come up with the idea of another table, and since we are always taking groups *of* a certain size, let's call this the "*of*" table.

Figure 14

"OF" TABLE

OF	O	I	N	M	&
O	O	O	O	O	O
I	O	I	N	M	&
N	O	N	&	II	IM
M	O	M	II	I&	NN
&	O	&	IM	NN	MI

As soon as the class has reasonably mastered the "of's", then we can give some thought to the reverse process, after illustrating the need with an example, such as "How many 'ughs' of steak for each of 'fo' families, in a dinosaur leg weighing 'tre fi un "ughs"'?" On our number scale, we can start from "oh" and repeatedly count "fo" until we reach "mi"; we can count in reverse fashion; we can refer to our "and" table, "anding" until we reach "mi"; or we can use the "of" table to find the answer to be "fo 'ughs'". Since this operation is almost always in answer to a question of the type, "How many a's in b?", let's call it the "in's" operation, and define the result as x in "x of $a=b$."

The next assignment might include such exercises as these:

15. Complete the "of" table from O to &.

16.

A.	IO OF &	IS ___.				
B.	N& OF II	IS ___.				
C.	IM& OF IIO	IS ___.				
D.	NOO OF ION	IS ___.				
E.	&I& OF MO&	IS ___.				
F.	HOW	MUCH	IS	&	OF	IO ?
G.	"	"	"	IN	OF	N& ?
H.	"	"	"	II	OF	II& ?
I.	"	"	"	INO	OF	NOO?
J.	"	"	"	MIN	OF	&I&O?

17.

A.	HOW	MANY	N	IN	II ?
B.			IO	IN	IOO?
C.			M	IN	MO?
D.			M	IN	MMM?
E.			N	IN	IIN ?
F.	HOW	MANY	&	IN	MI ?
G.			II	IN	INI ?
H.			NO	IN	&NO?
I.			IN	IN	IOM ?
J.			IM&	IN	INMMM? (ANS. &N)

18. Invent a symbol for "of".
19. Invent a symbol for "in" or "how many in".
20. Discuss the characteristics of the "fi" system in comparison with the decimal system in the repeated addition and subtraction processes. What apparent advantages does either have?
21. Devise an "of" and an "in" algorithm for *long* "of's" and "in's" (not using the present method of the decimal system).
22. Could every number in the "fi" system be represented by *a pair of numbers*?

Some of the obvious things that could or should have happened by now are:

a. discovery or verification (but not proof!) of *commutative*—

$$a+b=b+a \text{ and } ab=ba,$$

associative—

$$a+(b+c)=(a+b)+c \text{ and } a(bc)=(ab)c,$$

and *distributive* laws—

$$a(b+c)=ab+ac.$$

b. discussion of the properties of "oh" and "un" as identity elements of addition and multiplication.
c. discussion of the closed properties of right and left systems.
d. discussion of the effects of operations with mixed right and left numbers.
e. questioning by some student about the *in* operation where the result is not "even," e.g., found on the "of" table. This might create the need for a new system of numbers (perhaps "in" numbers would be appropriate), and probably we could continue on from there, if there seemed justification. Further work should probably be left to the curiosity and interest of individual students.
f. emphasis throughout on the idea of "agreements" being made, and on the important role "agreements" play in the development of a mathematical system:

As a result of these needs	*came these agreements*
1. expression of quantity	a method of counting
2. overcoming physical limits	counting by special groups of certain size
3. communicating over time and distance	words for numbers and groups
4. ease of writing and calculation	number symbols
5. larger numbers and ease of calculation	place value and zero
6. combining and reducing groups in quicker calculation	memorization of table (not exactly an agreement in the same sense as the others—but a necessary understanding between parties when rapid calculations are used)
7. multiplication and division	see agreement for 6
8. subtraction of numbers larger than minuend	extension of number system to negatives or "left" numbers
9. division that has no answer in multiplication table	extension of number system to fractions or "in" numbers
10. improved methods for more rapid calculation with large numbers.	accepted algorithms or methods for working with larger numbers.

g. a conviction that the "fi" system is more logical than the decimal system in its structure of name and value, and that it is simpler—requiring far less memorization. However, with larger numbers the number of symbols required is perhaps excessive (although the number of written letters is usually fewer).
h. the realization that all number systems have similar properties (there is nothing sacred about the decimal system), other ancient number systems being equally useful, and the modern binary and octal systems being *more* useful in special applications.
i. the frequent repetition of the question, "Why are we doing this?" A possible answer might be that biologists, doctors, and psychologists study animals to learn about men, and astronomers study stars to learn about the earth. Sometimes one can effectively learn about things close and familiar by studying the strange, the distant, and yet the similar. We don't want to fall into the trap of not seeing the forest for the trees, of taking things too much for granted. The teacher will, of course, mention applications of different number systems used in connection with modern computing methods.
j. challenges that can be issued such as:
   1. Devise new algorithms for the operations of the decimal system.
   2. Devise a number system of your own.
   3. Investigate the binary system, its applications, special properties, sample problems.
   4. Investigate how Romans may have performed the various operations, and devise algorithms for the common operations.
   5. Devise a fraction notation for the "fi" system. Discuss the properties of "ficimal" fractions and devise a notation for them.

I repeat, no claims or guarantees are offered. Can one become better at English by the study of a foreign language? Some will so maintain. I maintain only that the challenges to originality and pure intellectual exercise are few enough, that these ideas provide a great challenge, yet are within the capabilities of a great range of students and are not devoid of humor, and that the odds are pretty fair in favor of some real growth in appreciation of the evolution of our number systems and in a better understanding of them. If you agree with this proposition and want to try something that's fun and a little different, why not try the "fi" system?

New York City schools meet challenge

A significant study of New York City's June 1956 senior class of 27,756 young people (approximately 5% of the total U. S. school population) has been made by Samuel Schenberg, of the Board of Education. The investigation was undertaken to determine the extent to which high school students are studying science and mathematics, how many of those who plan to attend college hope to specialize in science or engineering, and whether homogeneous grouping of able students with an interest in these areas stimulates them to specialize.

His 32-page report indicates that 42% of the college-bound senior boys and girls in the academic and technical curricula wished to specialize in science. Girls chose pure science first and boys chose engineering, with pure science second. Extremely small percentages were interested in teaching science or mathematics. The data show a strong probablitity that the selection of careers as scientists and engineers increases in the ratio of 1:4:8 with the completion of two, three, and four years of science, respectively. In New York's three science high schools, containing 15.3% of all seniors enrolled in academic high schools, 93.2% of the boys and 98.5% of the girls planned to go to college, as against 87.5% and 78.3% in all other academic high schools. Of these, 82.0% of the boys expected to enter careers in science and 61.3% of the girls, as against 53.8% and 23.4%. The percentage of girls in the huge Bronx High School of Science who indicated an interest in specialization was nearly three times the percentage of girls showing such an interest in the city as a whole.

Mr. Schenberg offers five courses of action suggested by the data:

1. More comprehensive educational and vocational guidance programs, staffed with trained guidance counselors, and an adequate testing service to identify the most able students.
2. Homogeneous grouping of students with special interest in, and aptitudes for, science and mathematics, in special classes and special science high schools.
3. Effective guidance in the area of scientific careers for girls.
4. A minimum of three years of science and three years of mathematics for all college preparatory students.
5. A corps of "well-trained, highly professional and interested teachers of science and mathematics," with higher status and pay.—*Taken from* Engineering and Scientific Manpower Newsletter, *October* 25, 1957.

A note concerning the construction of $\sqrt{n}$

ROBERT A. OESTERLE, *Purdue University, Lafayette, Indiana.*

Irrational numbers deserve more attention than they usually get in high school. You may wish to try the methods suggested in this article.

A FAMILIAR TALE is that of the perifidious Pythagorean who disclosed the existence of the "unspeakable" numbers to the uninitiated and as a consequence was ostracized by the society. That perfidy is by no means confined to indiscretions of mathematicians during Pythagoras' time is evident as one considers the following definitions taken from a standard secondary school algebra text:

> *Integers* are whole numbers.
>
> An *integral expression* in algebra is one that does not have in it any fractions with literal numbers in the denominator.
>
> A *surd*, or *irrational number*, is the indicated root of a positive number of which the root cannot be expressed by an integer or integral expression.

Using *these* definitions a student logically concludes that

$$\sqrt{\frac{49}{25a^2}},$$

$a>0$ and rational, is an irrational number, since the principal root, $7/5a$, is neither an *integral expression* nor an *integer*. Attempting to utilize these definitions in relation to $\sqrt[3]{-29}$ is futile, since the third definition states explicitly that an irrational root is the "indicated root of a positive number . . . ."

The frequently seen and heard statement that irrational numbers are numbers whose roots cannot be found exactly is perhaps more fallacious than the quoted definitions, for any irrational number may be represented precisely and exactly by using symbols such as $\sqrt{3}$, $\sqrt[5]{-7}$, $\sqrt[3]{10}$, π, e, etc. Certainly the precision of such symbolism has little to recommend it in utilitarian applications, but the student should be aware of the fact that these are *exact representations*. Practical application generally demands an approximation of these values within the set of rational numbers, but such application in no way infringes upon the exactness of the original representation.

It is an encouraging trend, which is evident as one peruses the recent texts for high school algebra, that many authors have devoted a number of pages to the nature of numbers. The geometrical representation of the rationals as points on the number line is developed to a limited extent, but few writers carry the representation to the place necessary for the student to gain an intuitive realization that the set of rational numbers is everywhere dense. One text, however, does state:

> Early in your study of algebra you learned to represent rational numbers as points on a linear scale. But no matter how much you subdivide a unit on such a scale, points always remain for which there are no rational numbers.[1]

An anomaly enters the discussion at this point as the writers go on to assert:

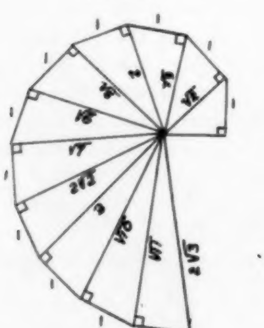
[1] Julius Freilich, *et al.*, *Algebra for Problem Solving, Book 2* (Boston: Houghton Mifflin Company, 1957), p. 199.

> It is possible to find a length which exactly represents an irrational number,[2] even though it is not possible to find the exact value of such a number.

More appropriately and correctly, this sentence should conclude with the clause: "... even though it is not possible to express the exact value as the quotient of two integers." The use of *exactness* to describe constructed line segments is of course assumed to imply theoretical exactness, since no measurement is exact.[3] The construction of a few of the irrational numbers is, however, a noteworthy feature of this text, and it provides the motivation for the following remarks concerning the construction of line segments that represent irrational numbers of the type $\sqrt{n}$, when n is a positive integer, and $\sqrt{p/q}$, when p and q are integers, $q \neq 0$.

The construction described by Freilich[4] depends upon successive utilization of previously constructed irrational numbers as shown in Figure 1. Obviously, if one wishes to construct a line segment of length $\sqrt{59}$, it is not necessary to begin with the isosceles right triangle with legs of unit length. A right triangle with legs of 7 and 3 may be constructed, and $\sqrt{59}$ constructed using the hypotenuse of this triangle.

Figure 1



[2] The use of "an" seems unfortunate, since most of the irrational numbers cannot be constructed using only ruler and compass. The classic example is $\sqrt[3]{2}$.

[3] Lawrence A. Ringenberg, *A Portrait of 2* (Washington: National Council of Teachers of Mathematics, 1956), p. 21.

[4] Freilich, *op. cit.*, p. 199.

Classically, the construction of $\sqrt{n}$ is accomplished as depicted in Figure 2. With diameter $n+1$, circle O is constructed. With $|AP| = n$ and $|PC| = 1$, PQ is constructed perpendicular to AC. Since AQC is a right triangle,

$$\frac{|QP|}{|n|} = \frac{1}{|QP|},$$

$|QP|^2 = n$, and $|QP| = \sqrt{n}$.

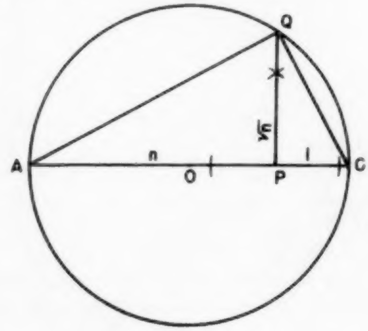


Figure 2

Certain minor disadvantages are inherent in these approaches. If one wishes to place the irrational numbers on the number axis, the method indicated in Figure 1 frequently necessitates successive constructions that multiply errors. The proof of the method shown in Figure 2 demands some knowledge of similar triangles, and as a consequence generally cannot be proved in the first course in algebra.

While no claim is made that the following construction is original, the writer has been unable to find it in standard texts and believes that it merits consideration. After successive constructions of right triangles with hypotenuse 2 units, leg 1 unit; hypotenuse 3, leg 2; hypotenuse 4, leg 3, etc., it becomes intuitively evident that the identity

$$\left(\frac{n+1}{2}\right)^2 - \left(\frac{n-1}{2}\right)^2 = n$$

is applicable to the proposed problem of constructing $\sqrt{n}$. Thus, to construct $\sqrt{n}$

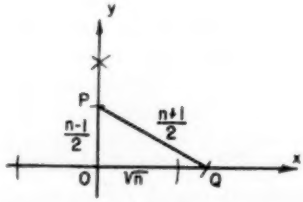


Figure 3

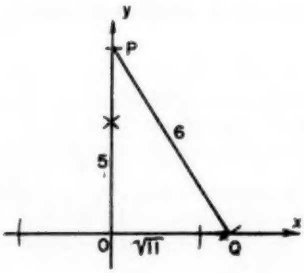


Figure 4

on the number axis, measure from the origin of the co-ordinate axis $(n-1)/2$ units on the y axis. The arc (Figure 3), with radius $(n+1)/2$ and center at P, intersects the x axis at the point Q; $|OQ| = \sqrt{n}$.

As an example, consider the construction of $\sqrt{11}$ as shown in Figure 4.

$$|OP| = \frac{n-1}{2} = 5$$

$$|PQ| = \frac{n+1}{2} = 6$$

$$|OQ| = \sqrt{\left(\frac{n+1}{2}\right)^2 - \left(\frac{n-1}{2}\right)^2}$$

$$= \sqrt{n} = \sqrt{11}$$

This technique is obviously extendable, since the identity holds for all n, to the construction of the square root of any positive rational number: $\sqrt{p/q}$, p and q integers, $q \neq 0$. In this case the identity may be transformed to

$$\left(\frac{p+q}{2q}\right)^2 - \left(\frac{p-q}{2q}\right)^2 = \frac{p}{q} \cdot$$

In Figure 5, to construct $\sqrt{13/7}$, let $p = 13$, $q = 7$.

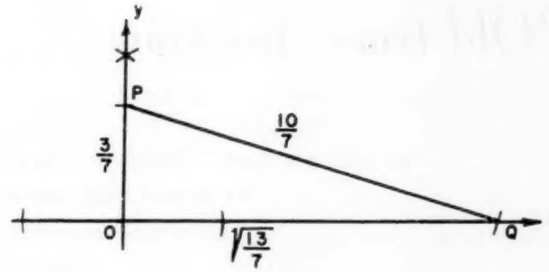


Figure 5

$$|OP| = \frac{p-q}{2q} = \frac{3}{7}$$

$$|PQ| = \frac{p+q}{2q} = \frac{10}{7}$$

$$|OQ| = \sqrt{\left(\frac{p+q}{2q}\right)^2 - \left(\frac{p-q}{2q}\right)^2}$$

$$= \sqrt{\frac{p}{q}} = \sqrt{\frac{13}{7}}$$

The discerning student will soon notice the existence of the subsidiary identity which pertains to Pythagorean problems of the type under consideration:

$$\frac{n+1}{2} + \frac{n-1}{2} = n$$

or

$$\frac{p+q}{2q} + \frac{p-q}{2q} = \frac{p}{q}, \quad q \neq 0.$$

Thus, given a right triangle with hypotenuse 48, leg 47; then

$$\frac{n+1}{2} = 48, \quad \frac{n-1}{2} = 47, \quad n = 95,$$

and the unknown leg has measure $\sqrt{95}$ units.

Pedagogically, the method of construction presented in this note provides: (1) an opportunity for students to observe or preferably discover an application of inductive reasoning within the scope of their mathematical maturity, (2) an illustration of the functional application of an identity, (3) a means of extending the students' understanding of the set of real numbers, and (4) an example of the interrelationship between algebra and geometry.

"Old tyme" fractions

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We should teach our students to search for patterns. This article describes how to conduct such a search in an interesting historical context.

THE USE OF FRACTIONS goes back to the very early mathematical records. The Egyptian scribe Ahmōse, when writing at about 1700 B.C., gave rules for their use that had been developed over centuries. The Egyptian idea of a fraction was based on the type $\frac{1}{n}$, or the nth part. By placing a special mark with their symbol for any integer, such as for 7, it then stood for $\frac{1}{7}$. These so-called unit fractions seem to have been used exclusively, the only known exceptions being a different way of treating $\frac{2}{3}$ and $\frac{3}{4}$.

We can only guess at the problems the Egyptian schoolboy was asked to do, but one such might have been the following: Given an integer n, to find two other integers x and y such that

$$\frac{1}{n}=\frac{1}{x}+\frac{1}{y}.$$

For $n=2$ we find that $\frac{1}{2}=\frac{1}{4}+\frac{1}{4}$. This is not the only solution, however, since it is also true that $\frac{1}{2}=\frac{1}{3}+\frac{1}{6}$.

For $n=3$ it turns out that $\frac{1}{3}=\frac{1}{6}+\frac{1}{6}$ and also that $\frac{1}{3}=\frac{1}{4}+\frac{1}{12}$.

The search for all possible representations of a given unit fraction as a sum of two such fractions makes an interesting problem. A solution was given in a note in *The Mathematics Student Journal*.[1] The author has found this problem particularly valuable for student research, since a minimum of skill is required to uncover a great variety of patterns and relations. Thus the reader has probably already noticed that

$$\frac{1}{2}=\frac{1}{4}+\frac{1}{4} \text{ and } \frac{1}{3}=\frac{1}{6}+\frac{1}{6}$$

are particular instances of

$$\frac{1}{n}\equiv\frac{1}{2n}+\frac{1}{2n};$$

and that

$$\frac{1}{2}=\frac{1}{3}+\frac{1}{6} \text{ and } \frac{1}{3}=\frac{1}{4}+\frac{1}{12}$$

can be obtained from

$$\frac{1}{n}\equiv\frac{1}{n+1}+\frac{1}{n(n+1)}.$$

[1] William E. Briggs, "On a Diophantine Problem," *The Mathematics Student Journal*, III (December 1956), 2.

The identity symbol of $\equiv$ is used because these equations hold for all values of n. (Except, of course, those that give a zero denominator.)

It is suggested that students determine all such expansions as above for the unit fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and on at least to $\frac{1}{10}$. This can be done by trial and error alone, with the main value to be gained from search for algebraic identities to add to the two given above. Here are some questions to stimulate the student's thinking:

1. In making an experimental search, just what combinations must be tried? Thus for $\frac{1}{3}$ the search for an integer succeeds with $\frac{1}{3}=\frac{1}{6}+\frac{1}{?}$, but fails for $\frac{1}{3}=\frac{1}{5}+\frac{1}{?}$; and succeeds again for $\frac{1}{3}=\frac{1}{4}+\frac{1}{?}$. We need look no further. Why?
2. How is the number of representations limited if n is a prime number?
3. If n is not a prime, how can the knowledge of its prime divisors be used?
4. Can you see why more than one identity can be constructed from which a certain result can be derived?

   Example: Compare $\frac{1}{6}=\frac{1}{10}+\frac{1}{15}$ with
$$\frac{1}{n(n+1)}\equiv\frac{1}{n(2n+1)}+\frac{1}{(n+1)(2n+1)}$$
   for $n=2$, and with
$$\frac{1}{m(m^2-1)}\equiv\frac{1}{m(m^2+m-1)}+\frac{1}{(m^2-1)(m^2+m-1)}$$
   for $m=2$.
5. Can you devise one or more general formulas which include all your results as special cases?

Given an integer n, then if integers x and y are sought such that
$$\frac{1}{n}=\frac{1}{x}+\frac{1}{y},$$
we may select x from the set $2n-k$ where $0\leqq k<n$. This follows, since if
$$\frac{1}{x}>\frac{1}{2}\left(\frac{1}{n}\right), \quad \text{then} \quad \frac{1}{y}<\frac{1}{2}\left(\frac{1}{n}\right).$$
A systematic search can, therefore, be based on the identity
$$\frac{1}{n}\equiv\frac{1}{2n-k}+\frac{n-k}{n(2n-k)},$$
where we demand that the numerator of the last fraction be an exact divisor of the denominator.

This reveals why only two representations are possible when n is a prime, for it can be shown that if p and k are integers and p is a prime, then $p-k$ divides $p(2p-k)$ if, and only if, $k=0$, or $k=p-1$.

It is often possible to learn most from our failures. Thus when we get $\frac{1}{3}=\frac{1}{5}+\frac{2}{15}$ we see that we can then write
$$\frac{1}{(2)(3)}=\frac{1}{(2)(5)}+\frac{1}{(3)(5)}.$$
The 2, 3, and 5 seem to be noteworthy: what we have is an instance of
$$\frac{1}{ab}\equiv\frac{1}{a(a+b)}+\frac{1}{b(a+b)}.$$
This last observation gives some insight into the structure of this problem, for if d is any integral divisor of n we have
$$\frac{1}{(d)\left(\frac{n}{d}\right)}\equiv\frac{1}{(d)\left(\frac{n}{d}+d\right)}+\frac{1}{\left(\frac{n}{d}\right)\left(\frac{n}{d}+d\right)}.$$

Some results, such as $\frac{1}{4}=\frac{1}{6}+\frac{1}{12}$, show all three denominators having a common divisor greater than unity. For these try

$$\frac{1}{(d)\left(\frac{n}{d}\right)(1)}\equiv\frac{1}{(d)\left(\frac{n}{d}+1\right)}+\frac{1}{(d)\left(\frac{n}{d}\right)\left(\frac{n}{d}+1\right)}.$$

The unlimited possibility for constructing identities is seen when one observes that

$$\frac{1}{ab}\equiv\frac{1}{a(a+b)}+\frac{1}{b(a+b)}$$

can be made an identity in one variable by setting b equal to *any* function of a. That is,

$$\frac{1}{(d)[f(d)]}\equiv\frac{1}{(d)[f(d)+d]}+\frac{1}{f(d)[f(d)+d]}.$$

Thus

$$\frac{1}{10}=\frac{1}{14}+\frac{1}{35},$$

or

$$\frac{1}{(2)(5)}=\frac{1}{(2)(7)}+\frac{1}{(5)(7)},$$

suggests

$$\frac{1}{n(2n+1)}\equiv\frac{1}{n(3n+1)}+\frac{1}{(2n+1)(3n+1)}$$

for $n=2$;

$$\frac{1}{m(m^2+1)}\equiv\frac{1}{m(m^2+m+1)}+\frac{1}{(m^2+1)(m^2+m+1)}$$

for $m=2$; etc.

Those students who enjoy working with algebraic patterns like these are likely to be interested in a branch of mathematics called the Theory of Numbers. Several excellent books on this subject have been published recently. They include Helen Griffin's *Elementary Theory of Numbers* (New York: McGraw-Hill, 1954), B. W. Jones's *The Theory of Numbers* (New York: Rinehart, 1955), Oystein Ore's *Number Theory and Its History* (New York: McGraw-Hill, 1948), and B. M. Stewart's *Theory of Numbers* (New York: Macmillan, 1952).

Alice in Wonderland

"... still the Queen kept crying 'Faster! Faster!', but Alice felt she could not go faster, though she had no breath left to say so. However fast they went they never seemed to pass anything.

" 'Well, in our country,' said Alice, still panting a little, 'you'd generally get to somewhere else—if you ran very fast for a long time as we've been doing.'

" 'A slow sort of country,' said the Queen. 'Now here, you see, it takes all the running you can do, to keep in the same place. If you want to get somewhere else, you must run at least twice as fast as that.' "

Editor's Note: Quoted because mathematics teachers these days can appreciate Alice's position.

Edited by Howard Eves, University of Maine, Orono, Maine

A history of computers, I

by Jules A. Larrivee, Lockheed Aircraft Corporation, Burbank, California

Introduction

Computing devices became necessary when men realized the need to *count* and *measure*. These two methods of computing form the basis for classifying computers as *digital* or *analog*. The digital computer operates on numbers expressed in some form of digital notation, and counts in discrete steps. The analog computer represents numbers in terms of some physical quantity such as length, voltage, pressure, etc., and operates continuously to obtain measurements which, when adjusted for proper scale factors, give the required result.

Examples of digital computers are the abacus, adding and calculating machines, and high-speed electronic digital computers. Graphs, charts, nomograms, slide rules, differential analyzers, and planimeters are analog computers.

Problems create the need for computers. Thus the abacus was developed to solve the arithmetical problems which arose in trade many centuries ago in Egypt, Greece, and the Far East. Planimeters were built to calculate the areas bounded by closed plane curves which could not be obtained by the methods of the calculus. Often computers introduce new problems or suggest new approaches to old ones. High-speed digital computers have made necessary a more profound study of the methods of numerical analysis, with correspondingly greater emphasis placed on a knowledge of the generation and propagation of errors.

The abacus

Originally, the abacus was an early device for tracing numbers on a dust-covered board. The earliest Greeks, in common with the Egyptians and Eastern peoples, used pebbles as counters in calculating (L. *calculus*, a pebble). Later, counters were made of metal, ivory, or colored glass, and were strung on wooden rods enclosed in a frame. A vertical rod was usually inserted, intersecting the horizontal rows of counters and separating them into two sets. In one form of the abacus there are two counters in one set and five in the other, the counters in the first set being equal in value to five times those in the other.

In China, a form of the abacus using bamboo rods in place of counters was known as early as the sixth century before Christ. The abacus in an improved form is still used in China under the name of swan-pan, and a similar instrument in Japan has the name of soroban. In the hands of a skilled operator, such a computer can equal and even surpass the performance of a modern desk calculator, as was shown in an experiment several years ago.

Logarithms and the slide rule

The invention of logarithms by John Napier (1550–1617), Baron of Merchiston, gave a great impetus to the art of computing. Later, Henry Briggs (1561–1631) modified Napier's original formulation and adapted logarithms to the base ten of our numerical scale. By such means, long and

tedious multiplications and divisions were reduced to additions and subtractions. Logarithms were of immeasurable assistance in astronomy and navigation; many of the basic relations in these subjects were rewritten to adapt them to logarithmic calculation.

Using the basic properties of logarithms, William Oughtred (1574–1660) invented the slide rule. His first rule was a circular one, the familiar rectilinear rule being first described in a work published in 1632. Many improvements have been made in the design and construction of slide rules, and today there are special scales and rules for particular types of problems. Within the limit of precision obtainable, the slide rule provides a rapid and easy means for performing the arithmetical operations required in the solution of scientific and engineering problems.

Adding and calculating machines

After the improvements in the abacus, no new digital device was invented until near the middle of the seventeenth century. In 1642, at the age of nineteen, Blaise Pascal (1623–1662) built an adding machine. The machine was mounted in a box, and horizontal wheels in front could be advanced one-tenth to nine-tenths of a complete turn. This movement was communicated to "figure-wheels," each bearing the numbers 0 to 9. A carrying device was provided so that the movement of a figure-wheel from 9 to 0 caused the next figure-wheel to the left to move through one-tenth of a revolution. The result was read through holes in the cover which displayed the uppermost figure on each figure-wheel. In England, in 1666, Sir Samuel Morland also built an adding machine, but without the tens-carrying device; the numbers to be carried were registered on small counter disks. In 1671, Gottfried Wilhelm von Leibniz (1646–1716) conceived the idea of a multiplying machine, regarding multiplication as repeated addition. His first complete machine was constructed in 1694. Leibniz introduced the "stepped-wheel," a wheel having teeth of varying lengths to represent the digits from 0 to 9.

During the eighteenth century there were many who worked on the design and construction of calculating machines. One of the main difficulties encountered was the high degree of accuracy necessary to make various parts, such as, for example, wheel teeth. Thus not only must the basic idea for such a machine be available, but the state of technology must be far enough advanced to produce parts of the required precision. As machines have become more complex, it has become necessary to employ specialists to fabricate the various components. And this has brought about a further complication, namely the need for outside capital. In the days of Pascal and Leibniz, one man could conceive such a machine, and design and build it himself. Nowadays the necessity for outside capital, among other things, has made such an undertaking impossible for one man.

The first successful calculating machine manufactured on a commercial scale was built by Charles Xavier Thomas of Colmar in Alsace in 1820. By setting pointers on a cover plate to one of the digits 0 to 9, a small pinion was caused to slide along an axle. Through a bevel wheel on the main shaft, this pinion was rotated through as many teeth as the digit set by the pointer. The amount of rotation was then transmitted to a "result figure-wheel" on a hinged plate which also carried a wheel indicating the number of turns made by the driving crank. A lever at the top permitted the selection of "addition and multiplication" or "subtraction and division." The manufacture of the "Arithmometer" by Arthur Burkhardt in 1878 marked the beginning of the German calculating machine industry.

The first change from the Leibniz stepped-wheel was made in 1872 by Frank S. Baldwin, who patented a machine in which use was made of a wheel from whose periphery a variable number of teeth (0 to 9) protruded. A more compact design was

made by W. T. Odhner, and machines of this type are available today.

The first key-driven machine was invented by D. D. Parmalee in 1850; and in 1887 Dorr Eugene Felt brought out the Comptometer, which is still widely used. E. D. Barbour combined a printing device with an adding machine, and about twenty years later, in 1890, the first practical adding and listing machines were introduced by D. E. Felt and W. S. Burroughs.

By incorporating a multiplication table in a calculating machine, Léon Bollée, in 1887, was able to perform multiplication in one operation rather than as repeated addition. The "Millionaire" calculating machine, manufactured by O. Steiger in 1893, made use of this idea.

In 1911, Jay S. Monroe and F. S. Baldwin brought out the first Monroe calculating machine. The original models were operated by a hand crank for adding and a hand lever for shifting in multiplication. The direction of rotation of the crank was reversed for subtraction. The introduction of electric drives in place of hand cranks has increased the speed of operation, and the modern version of the calculating machine in the United States is highly automatic. One recent model will perform the operation of taking square root automatically. Nonprinting calculators manufactured in the United States include Friden, Marchant, and Monroe.

The machines described above were designed and built with commercial applications in mind. Their use in the solution of scientific and engineering problems was a later development. Bookkeeping machines have several "registers," and it is possible to transfer information from one register to another. The possibility of using such machines for subtabulation and differencing was realized by L. J. Comrie of the British Nautical Almanac Office.

Planimeters and integraphs

One of the fundamental problems encountered in engineering practice is that of finding the area bounded by a closed plane curve. About 1815, a Bavarian engineer, J. H. Hermann, invented a device called a planimeter for performing this calculation. The rate of rotation of a wheel on a cone was determined by the distance of a pointer from a fixed center. The pointer was made to follow the curve. Later a disk replaced the cone, and many variations of the original design soon appeared.

The most popular planimeter was invented by Jacob Amsler in 1854, and is called the polar planimeter. An even simpler version is the hatchet planimeter of Captain Prytz, which he first built in 1857. Both ends of a bar are bent, one end being formed into a hatchetlike edge and the other into a point. The point is moved along the curve, dragging the hatchet edge as it moves. The difference in direction in which the bar points before and after making the circuit of the curve gives a measure of the area.

Abdank Abakanowicz in 1874, and C. V. Boys in 1890, independently invented the integraph, a device which draws the integral of an arbitrary function plotted to a suitable scale on paper. This device can be used to solve some types of differential equations.

Differential analyzers

To solve differential equations by mechanical or electrical means, it is necessary to have an equivalent of the process of integration. Adopting an idea first proposed by J. J. Thomson, Vannevar Bush in 1927 constructed an integrator using a disk rolling on a wheel. The "Differential Analyzer," as the completed machine was called, could solve nonlinear as well as linear differential equations, and through proper use of integrators could be used to generate functions by means of the differential equations which define them. Auxiliary equipment was built in, such as input and output tables, and special gear boxes; later developments included the addition of electrical controls.

The variables occurring in the problems were represented by shaft rotations.

The mechanical differential analyzer has been largely supplanted by the electronic differential analyzer. The basic component is the operational amplifier which, when properly connected with resistors and capacitors, can serve as an integrator, an adder, or a sign changer. The variables occurring in the problems are represented by voltages. By the use of potentiometers, multiplication by a constant can be effected. For the multiplication of variable quantities, use is made either of servo multipliers or electronic multipliers. Limiters are used for the representation of discontinuous functions, such as coulomb friction. Function generators of many types are available. One of the accessories is a curve plotter which automatically traces the solution curve on graph paper. The accuracy of such a computer is clearly limited by the tolerances of its component parts. Computers of this type have been used in the aircraft industry for a long time in the study of vibration and flutter, and as simulators. They have also been used in the study of control systems and as adjuncts to high-speed digital computers.

Other analog computers

Formulating differential equations in terms of finite difference approximations has been the starting point for building resistance networks to serve as analog computers for finding solutions to such equations. G. Liebman, in England, has demonstrated the usefulness of such computers in solving boundary-value problems in mathematical physics. A similar concept using a network of resistors and capacitors led to the building of the Heat and Mass Flow Analyzer by V. Paschkis of Columbia University. By means of such a network, thermal problems, such as the heat flow through an insulated pipe, have been solved in a matter of seconds. If the measurements of heat flow had been made on the actual physical system, it would have taken days or months to complete the solution of the problem.

Electrolytic tanks, conducting sheets, membranes, and sand-heaps are among the many other types of analog computers which have been constructed for the purpose of solving particular problems. Many special devices have been built to serve as function generators, as delay-time simulators, and to represent discontinuous functions of various types.

Network analyzers, so-called because they were originally built for solving problems in power transmission, have found wider application in the solution of linear simultaneous algebraic equations. Linear equation solvers are available commercially.

Problems originally thought solvable by analog methods alone have since been found to be adaptable to digital methods. This change has been brought about by higher speeds in digital computers and partly by advances in the methods of numerical analysis.

Babbage

About 1812, Charles Babbage (1792–1871), an Englishman, started work on the project of building an automatic computer. The idea was perhaps suggested to him by watching the operation of a Jacquard loom where the pattern to be woven is controlled by means of holes punched in cards.

Babbage's first venture was the construction of a "Difference Engine." If a series of values in consecutive order is taken from a table of sines or logarithms and differenced by subtracting each value from the one immediately following it, another series of values is obtained called the first differences of the original values. If this procedure is applied to the set of first differences, a set of second differences is obtained, and so on. It will be found that for functions such as sines, cosines, and logarithms, differences of sufficiently high order are constant or nearly so. Babbage's idea was to start from the

constant differences and by repeated addition reproduce the function values. By 1822 he had constructed a working model. At about this time the British government agreed to finance the building of a full-sized difference engine. Not only did Babbage have to design the engine itself, but often the tools needed to make the parts. Though Babbage lived at the beginning of the Industrial Revolution, the consummation of his ideas in the high-speed electronic digital computer may be said to have ushered in a Second Industrial Revolution.

The Difference Engine in no way represented the extent of Babbage's imagination. He conceived of a machine which he called an "Analytic Engine," which, if it had been built, would have included many of the most modern ideas in digital computers. In addition to being able to add and print, the computer was to multiply, divide, call for new data from the operator, and do many of the things that a human operator would be called upon to do. The input was to be made by punched cards, and Babbage devised a code in which to express not only the numbers to be operated upon but also the instructions to the computer.

The Analytical Engine was never completed, partly because of difficulties encountered in the making of high precision parts and partly because of the withdrawal of financial support by the British government. At least two complete different engines were built. One of these, built by George Scheutz and Son of Sweden, was originally located at the Dudley Observatory in Albany. Later it was acquired by Felt and Tarrant, and is now on display at 1735 North Pauling Street, Chicago.

Punched card computers

Babbage's ideas were not put to use until many years after the original venture failed. In 1890, Herman Hollerith, an employee of the Census Bureau, invented a tabulating machine operating by means of holes punched in cards. The presence or absence of a hole in a particular location on a card was tested by a sensing device which in turn operated a clutch. The punched card computers made by International Business Machines and Remington Rand are adaptations of the basic Hollerith principle.

[*To be concluded in the next issue.*]

An "a" for an "i"

by Rev. Dare Morgan, S.J., St. Ignatius High School, San Francisco, California

The intensified mathematics program in high school will introduce many students to the great mathematician of the seventeenth century, Francesco Cavalieri (1598–1647), whose principle of indivisibles is one of the important gateways leading to the integral calculus. In the printed account of Cavalieri's life there is a persistent error—not of a mathematical nature—that appears in some of the major encyclopedias[1] and source books,[2] with one or two exceptions.[3] It is the statement that Cavalieri was a Jesuit. The truth is that

[1] *Encyclopaedia Brittanica*, vol. V, p. 66; *Encyclopedia Americana*, vol. VI, p. 137.

[2] F. Cajori, *History of Mathematics* (New York: Macmillan Co., 1938), p. 160; D. E. Smith, *History of Mathematics* (Boston: Ginn & Co., 1923), p. 362.

[3] D. E. Smith, *Source Book of Mathematics* (New York: McGraw-Hill Book Co., 2d impression, 1929), p. 605; *Catholic Encyclopedia* (New York: Robert Appleton Co., 1908), vol. III, p. 468.

Cavalieri was not a Jesuit but a *Jesuat.*

The religious Congregation known as the Jesuats—which was in no way related to the Jesuit Order—was founded by Blessed John Colombini of Siena in the fourteenth century and approved by Pope Urban V in 1367. Their original work included the care of the sick, especially the plague-stricken, and the burial of the dead.[4] However, with the passage of years the Jesuats diminished in numbers to the point where an attempt at revival was made necessary in 1606.[5] A few years later Cavalieri, at the age of fifteen, was received as a member. After completing his studies, he taught theology for a short time, but his talent for mathematics was soon recognized and he was permitted to study under Castelli, a pupil of Galileo. He then spent the remaining eighteen years of his life teaching mathematics in the University of Bologna.[6]

Unfortunately the Jesuats as an organization did not enjoy the success that their gifted son acquired. Over the years some abuses had made their way into the group. As a result, today it no longer exists. The manufacture and sale of distilled liquors[7]—presumably in a way contrary to Canon Law—together with a scarcity of members,[8] have been some of the reasons given by writers for the group's suppression by Pope Clement IX in 1668.[9]

[4] *Cath. Encycl.*, vol. VIII, p. 458.

[5] *Butler's Lives of the Saints*, ed. H. Thurston, S. J., and D. Attwater (New York: P. J. Kenedy & Sons, 1956), vol. III, p. 228–30.

[6] *Cath. Encycl.*, vol. III, *ibid.*

[7] W. E. Addis and T. Arnold, *Catholic Dictionary* (15th ed.; Catholic Publication Society, 1951), p. 466.

[8] Butler, *ibid.*

[9] *Cath. Encycl.*, vol. VIII, *ibid.*

Letters to the editor

Dear Sir:

The enclosed on the Euler Identity may be of interest to you as a filler for THE MATHEMATICS TEACHER.

This magazine has only recently come to my attention. It seems altogether admirable to me; so keep up the good work and convey my warm appreciation to your colleagues.

Yours sincerely,
John Coleman
University of Toronto

Professor Yates's interesting proof ("The Euler Identity: $e^{ix} = \cos x + i \cdot \sin x$" in THE MATHEMATICS TEACHER, April 1958) can be simplified as follows:

Let $u = (\cos x + i \sin x)e^{-ix}$, then it follows immediately that $u' = 0$, but $u(0) = 1$. Therefore $u(x) = 1$. q.e.d.

A similar argument enables one to give an elementary proof of the important fact that the solution of the equation $y' = ky$ is unique to within a multiplicative constant. For suppose that u and v are two solutions of this equation, then uv^{-1} has zero derivative and is therefore a constant. For some years I have used the Euler identity as an interesting application of this uniqueness theorem in my introductory calculus course at Toronto. The students usually react to the imaginary constant with intrigued skepticism!—*A. J. Coleman, Department of Mathematics, University of Toronto.*

Dear Sir:

I would like to express my protest at such a large increase in the cost of membership in the Council and of its journals. It would seem that if anyone should realize the importance of holding the line against inflation and all its evils, it should be the supposedly objective-minded teachers of mathematics. Moreover, I find no such comparable increase in my teaching salary. This is a real disappointment to a long-time member.

Now that's off my chest, may I make a combination suggestion-request? I'd like very much to see more articles of down-to-earth, high school level material about the new math we are being urged to teach. Can't we have some simple, elemental articles about sets, etc., and how to teach this material that most of us teachers never heard of when we were in school? Personally, I'd much prefer that to a lot of the flights of fancy that I seem to find in THE MATHEMATICS TEACHER of late.

Respectfully,
PAUL L. TAYLOR

Edited by Lucien B. Kinney, Stanford University, and Dan T. Dawson, Stanford University, Stanford, California

A gratuitous extrapolation in the teaching of per cent

by Francis J. Mueller, Maryland State Teachers College, Towson, Maryland

Throughout the teaching of mathematics there is a tendency to overestimate the degree of transfer of learning. In our student days we have all had our frustrating moments with the likes of ". . . the remainder of the proof is obvious and will be left as an exercise for the student."

In the teaching of arithmetic, perhaps one of the most frequent and flagrant of such "gratuitous extrapolations" occurs in the handling of per cents greater than 100%.

Having taught the student a variety of percentage problems and their means of solution, teachers all too often fail to make specific mention of the important distinction between such statements as "175% of something" and "175% greater than something." Among the college students that I teach there appears to be an ingrained notion that no difference exists between these two statements, that they are quite interchangeable.

From what I can discern, my students—and I have reason to believe they are not anomalous—were exposed to their share of work with percentage problems in their precollege training, both the per-cents-less-than-100% and the per-cents-greater-than-100% varieties. However, it would appear from the students' reactions that their teachers must have *assumed* they understood that "175% greater than" is obviously equivalent not to "175% of," but to "275% of."

With some, the confusion extends over into the area of per-cents-less-than-100%, e.g., "75% of" = "75% greater than." Of course this is not so frequent, for generally the student has by this time recognized that "75% of something" is not going to yield him an amount greater than he originally had. But with per-cents-greater-than he has no such simple check on his reasoning, since both "175% of" and "175% greater than" yield him an amount greater than he originally had.

Since we know that these distinctions are sources of potential confusion for the student, we will do well to point out that fact and show the way to correct interpretations. This might be accomplished in three phases.

In the first phase, the student must be made consciously to understand the difference between the two basic comparative statements: "so many times as much as" and "so many times more than." Use of hypothetical, concrete situations can be most helpful here: John has $12 and Mary has $4. Have the students make statements comparing (by ratio) John's wealth to Mary's. Most will give as a first reply, "John has three times as much money as Mary." If an alternative statement involving "greater than" is not forthcoming, it can be drawn out usually by asking

questions like: "How many dollars more than Mary does John have?" (Answer: \$8.) "How does this *excess of dollars* which John has compare to what Mary has?" (Answer: 2 to 1.) "Involve this ratio in a statement of the situation." (Answer: John has two times more dollars than Mary.)

Given practice with additional similar illustrations, the students soon reach a point at which they can readily express a comparison either in terms of the absolute values involved or in terms of the excess of one over the other—and appreciate the difference in the statements.

The second phase involves translating these alternative statements for each comparison to expressions of per cent: If John has three times Mary's dollars, then John has 300% as much money as Mary; if John has two times more dollars than Mary, then John has 200% more dollars than Mary; and the like. Then, upon abstracting the essentials from these equivalent alternatives, the student sees that:

$$200\% \textit{ more than} = 300\% \textit{ of}$$
$$100\% \textit{ more than} = 200\% \textit{ of}$$
$$50\% \textit{ greater than} = 150\% \textit{ of}$$
$$\textit{an increase of } 75\% = 175\% \textit{ of}$$

The final phase calls for directing *specific* attention to the fact that the basic percentage problems (whether by the three-case method or formula method) as the student learned them were all beholden to the "of" language. Instead of developing a separate technique for these "excess" cases, the simplest thing to do is to translate instances of "per-cents-greater-than" to the "of" language by the simple expedient of adding 100% and proceeding in the usual way.

Before leaving the topic, however, the good teacher will not take it for granted that the student surely understands that a similar treatment can be made for "per-cents-less-than." Instead, he will develop the parallel notions:

$$40\% \textit{ less than} = 60\% \textit{ of}$$
$$22\% \textit{ less than} = 78\% \textit{ of}$$
$$\textit{a decrease of } 35\% = 65\% \textit{ of}$$
$$x\% \textit{ less than} = (100-x)\% \textit{ of}$$
$$120\% \textit{ less than} = \textit{an absurdity}$$

Now, having followed this suggested pattern in your classroom, certain that the light in the students' eyes reflects crystal-clear understanding, I suggest you try them, without notice, on a simple discount problem, such as: "A suit, priced at \$75, is offered to you at a discount of 18%. If you accept, what price would you pay?"

Check to see how students got their answer of \$61.50. For those students who multiplied \$75 by .82, you may permit yourself a well-earned glow; for those who, despite your good instruction, methodically took 18% of \$75 and then subtracted, don't despair—they have simply re-emphasized for you that the only way to assure transfer is specifically to teach for it.

A request from the subcommittee on junior high school mathematics of the Secondary School Curriculum Committee

Teachers and schools who are conducting experimental programs in mathematics at the junior high school level are requested to report their plans and experiences to the subcommittee on junior high school mathematics of the Secondary School Curriculum Committee of the National Council of Teachers of Mathematics. These reports should be mailed to Professor John A. Brown, College of Education, University of Delaware, Newark, Delaware. The subcommittee is developing a statement on current practices in junior high school mathematics and on recommendations for the immediate future. Your cooperation through suggestions and reports will be appreciated.

• **POINTS AND VIEWPOINTS**

A column of unofficial comment

The weight of a symbol

by Harold P. Fawcett, President of The National Council of Teachers of Mathematics

On the wall above my desk is a gavel. It was handed to me by Howard Fehr, the immediate past-president of The National Council of Teachers of Mathematics. During this brief and simple ceremony he referred to the "weight" of the gavel, and I have since looked at it many times with this significant reference in mind. The handle is not long, other dimensions are in like ratio, and the hasty observations of an uninformed eye might well suggest that this particular gavel is not very heavy. But the accuracy of such a conclusion depends altogether on what one sees as he looks at this symbol of authority hanging on the wall of my study. Let us look at it together and then make some estimate of its weight and its influence in shaping the world of tomorrow.

This gavel was transferred to me in Cleveland at the closing luncheon of the Annual Meeting on April 12, 1958. Among the many luncheon guests there were undoubtedly some who saw in the process little more than a small piece of wood nicely machined and beautifully finished. But seated in that great audience were others who looked at this gavel and recognized the creative vision of those 127 dedicated men and women who just thirty-eight years ago had gathered in this same Ohio city for no other purpose than to plan together how the teaching of mathematics might be improved. It is out of such considerations that The National Council of Teachers of Mathematics was born, and of the original 127 who created this organization, the three who were present at this Saturday luncheon almost forty years later must indeed have looked in proud perspective at the achievement of the intervening years. The gavel which closed that first historic meeting has steadily increased its "weight," and now, after nearly two decades of service, The National Council of Teachers of Mathematics reflects the projected wisdom of those distinguished men and women who created it.

In many a classroom the teaching of mathematics has come to life through the wholesome influence of Council publications. The three journals continue to awaken the untapped potentialities of both teachers and students. The series of small publications has helped to transform dull and uninteresting classrooms into centers of purposeful activity. The twenty-three yearbooks provide the finest available library of professional literature dealing with mathematics education, and those now being prepared for publication will add to the prestige of this collection. Beginning with the initial group of 127, the membership has now increased to approximately 20,000, and the gavel of the Council has indeed taken on a lot of added "weight."

Let no one believe, however, that this is any time for a reducing diet, popular though this practice may at present be. During the thirty-eight-year history of The

National Council there has never been a time when its increasing "weight" was more needed than at present. Curricular practices have been questioned and are now under serious study. Well-conceived experiments are testing the educational values inherent in new content, and research of this type should continue through the coming years. On other fronts, proposals are developing, designed to breathe a new spirit into older and more familiar content so that it may effectively serve the changing needs of society. Curricula are being reorganized, plans to provide new teaching materials are under way, and it is not surprising that all of this ferment leads to some degree of unrest among mathematics teachers.

The Secondary School Curriculum Committee of the Council is thoroughly informed concerning all of these developments, and is now preparing a preliminary report based on questions which have been raised by teachers and administrators who wish to improve the quality of mathematics teaching in their respective locations. Individual members of this committee are working intimately with other groups, and it is anticipated that the final committee report will reflect the best of all that is known concerning the kind of mathematics curriculum which will most effectively serve the needs of all secondary school students.

When the secondary schools of the United States open tomorrow morning, approximately 65,000 mathematics teachers will report for duty. Considerably more than one-quarter of them are members of the National Council, and others are joining in increasing numbers. None are more dedicated to the high responsibilities of their profession and none are more faithful in meeting them.

It must not be overlooked, however, that many of these teachers are operating on emergency certificates and that this number is likely to increase. To estimate the percentage not regularly certified is virtually impossible, but in one state twelve per cent are reported "as being without certificates," while in another only twenty-one per cent are reported as having majored in mathematics. Then there are all too many others whose only qualification for teaching mathematics is that they have a free period at the scheduled hour. The assumption frequently made by worried administrators that *anyone* can teach mathematics should be abandoned. Music teachers, coaches, home economics majors, and physical education experts, able and competent in their respective fields, may also be "good" in mathematics but, in general, they are not good enough to be drafted as teachers in this important area. The spotlight should also be turned on the mathematical background of those certified to teach in the elementary school, since the curriculum choices made in the high school are so largely influenced by the attitudes and interests developed in earlier years.

Of the many important problems related to the teaching of mathematics two only have been specifically mentioned. The National Council of Teachers of Mathematics is concerned with all of them, and through its thirty-two committees is working on many fronts. The voice of the Council is the voice of 20,000 mathematics teachers, and every single member (including those who are married) is adding "weight" to the symbolic gavel hanging on my study wall. It is, indeed, heavy, as Professor Fehr stated, and it must be kept heavy. It symbolizes an increasing membership of dedicated teachers, unified in their support of the basic purposes defined thirty-eight years ago by those 127 charter members who wished "to plan together how the teaching of mathematics may be improved." To serve as an instrument in moving toward the achievement of this worthy purpose is a most rewarding experience.

Reviews and evaluations

Edited by Richard D. Crumley, Iowa State Teachers College, Cedar Falls, Iowa

BOOKS

Analytic Geometry, Edwin J. Purcell (New York: Appleton-Century-Crofts, Inc., 1958). Cloth, x+289 pp., $4.50.

In favor of the present volume is its easily readable style and clear exposition, as well as its inclusion of numerous interesting topics and problems. A chapter on curve sketching, for example, is a valuable feature.

Many users will find the pace of the book somewhat too leisurely and unchallenging, and language purists may perhaps object to such statements as that on page 146: "The locus . . . is symmetric with respect to the origin if substitution of $-x$ for x and $-y$ for y throughout the equation leaves the equation essentially unchanged."

The main reservations concerning this book probably center around two other points.

First, no use is made of vectors. This involves several drawbacks, all severe. Most important is the fact that the student is denied the use of valuable mental pictures. At a time when various authors are making effective use of geometrical mental pictures even in such unlikely subjects as linear transformations, probability, and complex variables, it is hard to justify taking a genuinely geometric subject and depriving it of its means of visualization. Moreover, the uses of analytic geometry in physics and engineering are nowadays founded mainly on vector formulations. Finally, it is the vector formulation which extends to more advanced work in tensor calculus, differential geometry, and linear spaces. Consequently, the failure to present analytic geometry in a vector formulation puts obstacles in the way of the student's present understanding and requires him to relearn the subject later if he continues his studies.

The second reservation that one feels about this volume is that it is not a book for 1958. The movement today is to combine subjects in a unified way wherever possible—or, one might say, to eliminate old and artificial compartments that obscure the true unity of mathematics. The new place for analytic geometry appears to be in combination with synthetic geometry at the high school level, and in combination with calculus at the college level. This book, despite its virtues, does not in any way represent a real step forward.—***Robert B. Davis**, Syracuse University, Syracuse, New York.*

College Plane Geometry, Edwin M. Hemmerling (New York: John Wiley and Sons, Inc., 1958). Cloth, ix+310 pp., $4.95.

This is a disappointing book. It claims to be a college text, and its preface states that geometry for colleges " . . . must differ in many respects from that taught in high schools." Unfortunately, these differences cannot be said to have had a beneficial influence on the mathematical content of the book. The author seems to have gone out of his way to retain in his treatment many of the worst features of many of the worst high school books on the subject.

The student is urged to improve his thinking habits and is promised lessons in clear and critical thinking. The text itself furnishes repeated examples of the very opposite. For instance, the definition of congruence given is, "Two figures are congruent when they have the same size and shape." The standards of precision employed in the proofs are also of a very low order. The book is open to further severe criticism on the basis of the selection of topics. In fact, nowhere in the treatment of these matters is there any evidence to be discerned that the author is familiar with geometry as mathematicians have understood the subject for the last seventy-five years. The whole point of view seems to be that a snappy format in a text licenses publisher and author to ignore major intellectual and professional responsibilities.—***Howard Levi**, Columbia University, New York City, New York.*

Essential Mathematics for College Students, Francis J. Mueller (Englewood Cliffs, New Jersey: Prentice-Hall., Inc., 1958). Paper, xiii+238 pp., $3.95.

The author states that this manual has been prepared with three kinds of users in mind: those who want to acquire or renew a basic mathematical competence sufficient to cope with everyday problems; those who must prepare themselves to meet the mathematical demands of nonmathematical courses; and those who need additional preparation or review before pursuing more advanced courses in mathematics. In view of the rather extensive emphasis placed upon the restudying of arithmetic as covered in the first seventy-six pages of the manual, it seems that this book should be used in a beginning algebra class in high school or in a college algebra class designed for those who have had no algebra in high school. Another

possibility would be use in a college class consisting of those students who have had some training in algebra in high school, but who show a lack of proficiency in it when they enter college.

The material concerning arithmetic covers the usual topics, with the addition of some work concerning approximate numbers. The explanations given are clearly written and complete. The number of examples included, together with detailed explanations of solution, is sufficient to give the student a very adequate understanding of how to perform an operation.

In making the transition from arithmetic to algebra, the author attempts to correlate the two as much as possible, yet he points out some of the differences. If students have had no experience with signed numbers prior to encountering them in this book, the teacher would need to supplement the material quite extensively. Signed numbers are treated from a purely operational point of view, with very little effort to make these operations meaningful.

The section dealing with the solving of verbal problems is excellent. The technique of translating the problem from everyday English to algebraic English to an algebraic equation is used. This gives the student an opportunity to become acquainted with a new way of stating a relationship before he is expected to write the algebraic equation. The numerical approach to the solving of a verbal problem is also emphasized.

Throughout most of the work in algebra, an effort is made to help the student develop the reasons why the rules work instead of merely giving him those rules. The statistical work is very elementary in nature and includes concepts which are likely to be encountered in everyday reading.

It seems to the reviewer that this book would have greater possibilities for high school algebra courses than it would for college courses. It could be used for remedial work in college either on a class basis or on an individual basis. This book would seem to be good for use with those who are rather slow learners.—*Harold W. Brockman, Capital University, Columbus, Ohio.*

Handbook of Calculus, Difference and Differential Equations, Edward J. Cogan and Robert Z. Norman (Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1958). Cloth, xii+263 pp., $4.50.

In addition to the usual tables of values and lists of formulas for elementary functions, this handbook contains lists of "general solution forms" for certain elementary differential and difference equations. It is hoped (by the Committee on the Undergraduate Program of the Mathematical Association of America, among others) that this method of "teaching" differential equations will save time needed for more important topics. The problem of presenting the topic of elementary differential equations concerns very few teachers of high school mathematics. Even the most idealistic students of the curriculum have not suggested that this topic be taught in high school. This reviewer feels, therefore, that this book has little of interest for teachers of high school mathematics.—*M. F. Smiley, State University of Iowa, Iowa City, Iowa.*

Improving the Arithmetic Program, Leo J. Brueckner (New York: Appleton-Century-Crofts, Inc., 1957). Paper, vii+120 pp., $1.25.

In the introductory chapter of this monograph, Dr. Brueckner states that its major purpose is to assist schools in evaluating and improving their arithmetic programs. He justifies its publication by asserting that keeping the arithmetic program of a school in line with the needs of our changing society and keeping the instructional procedures in line with the findings of research are constant problems facing those persons who are responsible for guiding and directing the program. He points out that citizens of the community as well as teachers, principals, superintendents, curriculum directors, and specialists should participate in improvement studies. These should involve gathering information concerning the needs of the children and the community, analyzing the strengths and weaknesses of the existing program, and outlining immediate and long-range steps for improving the program. A copy of the guide for the study of the arithmetic program, developed under the direction of the author in courses and workshops at the University of Minnesota, is included. This guide was developed in connection with the activities of the Minnesota State Committee for the Improvement of Education in 1954–55. The major sections of the guide deal with the (1) objectives and outcomes of the arithmetic program, (2) organization and content of the curriculum, (3) instructional practices and procedures, and (4) availability and adequacy of instructional materials. The subsequent chapters of this monograph contain background material as well as specific suggestions for studying, evaluating, and improving the arithmetic program in these areas.

School administrators who are making plans to study the arithmetic programs of their schools will find many suggestions here for organizing and guiding such a study. Classroom teachers and members of the community who are to participate in the study will have a clearer idea of what they are attempting to do if they read this book first. The emphasis given to the study of the objectives of the instructional program and the suggestions to re-evaluate each objective and estimate the degree to which each is being achieved should deter those administrators and teachers who are inclined to evaluate school programs in piecemeal fashion. The stress that is placed on the need for careful collection and critical appraisal of data should convince those who are inclined to move rapidly that more than a hasty look is necessary before proper steps can be taken to improve a program.

Examination of this monograph should not be limited to those persons who are only concerned with improving the program in arithmetic. The suggestions found here will be of help in the study of any area of the curriculum, especially with regard to (1) planning and carrying through a curriculum development program, (2) evaluating appraisal procedures, particularly the standardized testing program, (3) studying and evaluating instructional practices and procedures, and (4) developing a plan for improving instruction.

In the opinion of the reviewer, insufficient attention is given to the role of the consultant and to the amount of time needed to do such a comprehensive study. Too frequently, school administrators directing such studies expect the consultant to take the leadership role and fail to give teacher and citizen committees sufficient time to do thoroughly and well what they have been asked to do. The bibliography seems to be lacking in some respects, considering the wide range of background and experience of the people who will be using this monograph. These shortcomings are relatively minor, however, and do not limit the value of this publication to a very measurable extent. The author has a rich background of experience from which he has drawn to make this a practical guide. Professional and lay people earnestly endeavoring to study and improve the arithmetic programs of their schools who use this guide should be able to identify with confidence points at which changes are desirable.—*Joyce Benbrook, University of Houston, Houston, Texas.*

An Introduction to the Foundations and Fundamental Concepts of Mathematics, Howard Eves and Carroll V. Newsom (New York: Rinehart and Company, 1958). Cloth, xv+363 pp., $6.75.

Here is a book with special appeal to many readers of The Mathematics Teacher. It develops the history of many of the important aspects of "modern mathematics"—including most of those now widely advocated for the high school curriculum—by a method which is not "mere history" but a historically-enriched discussion of ideas and mathematical trends. Geometries, including non-Euclidean ones, algebraic structures, sets, and symbolic logic are some of the topics treated. A mature reader who has forgotten much of his technical knowledge of advanced mathematics can gain many insights from this book, which is intended primarily for advanced undergraduates. Some readers will regret that after a careful introduction of many modern topics the authors are forced to slight the detailed development of theorems and applications. Still, those developments can be found in many other books, while here we find stimulating answers to the questions: What are the distinctive viewpoints of modern mathematics? How and why did they come about? With what rival views do they compete?

Postulational thinking is called "The Modern Mathematical Method," the title given to Chapter 6. That there are unsolved problems concerning some aspects of this method is indicated by the comments on logicism (p. 287) and on formalism (p. 290f.) and even by the closing footnote, a semihumorous quotation on mathematics as a branch of theology.—*Carl H. Denbow, Ohio University, Athens, Ohio.*

Mathematical Excursions (Dover republication), Helen A. Merrill (New York: Dover Publications, Inc., 1957). Paper, xi+145 pp., $1.00.

Twelve of the fourteen chapters of this booklet are devoted to arithmetical pastimes ranging from criteria for divisibility to applications of Diophantine analysis. The last two chapters are concerned with linkages and some unsolved problems in number theory. Explanations are simple and, for the most part, adequate.

Since this is a set of "excursions," it cannot be expected to compare with standard books on mathematical recreations, but must be considered on its own merits. Thus considered, each chapter turns out to be a very short trip into an area not usually covered in the classroom, no two trips being related. This format may well serve to whet the mathematical appetite of the more able student in the secondary school mathematics class.

It is unfortunate that this is merely a reprint of the original, published in 1933. As such, it is somewhat dated. For example, Chapter 2 discusses the "dyadic" notation. We know this better as the binary notation. Again, the reprint includes all the faults of the original, as well as its merits. Thus, some of the chapters are followed by problems for solution, whereas others are not. Also, the space devoted to some of the exposition is often disproportionately allotted. For example, three pages are used to explain the formation of odd magic squares by De la Loubère's method, which could be done in one paragraph, while half a page is used to show how to solve a linear Diophantine equation. The "fifteen" puzzle is presented without any discussion of possible solutions, and determinants are introduced without any application at all.

As suggested, this booklet could be used to interest the better student as a form of enrichment.—*Samuel L. Greitzer, Bronx High School of Science, New York, New York.*

Mathematics in Business, Lloyd L. Lowenstein (New York: John Wiley and Sons, Inc., 1958). Cloth, xv+364 pp., $4.95.

The reviewer feels that this textbook fills a definite need for a book in the field of business mathematics which is between the usual business arithmetic and the usual mathematics of finance textbooks in difficulty and in scope of topics

covered. As such, it is a welcome addition to the market. It is well written in general, and the problem material is challenging, particularly in the portions directed to applications of the principles developed.

General annuities are not considered nor are the usual topics of life insurance. This is probably desirable, since the group for whom the textbook is intended are usually deficient in background training in mathematics, and as a result cannot be expected to perfect their understandings of the concepts of algebra required for a treatment of such topics in one short course in the mathematics of business. The usual "identity" type of problem is also lacking. Since such problems require an excellent understanding of algebraic principles, their omission is understandable. It is the feeling of the reviewer, however, that a certain amount of such material would be highly desirable in building a deeper understanding of the interrelationship between the concepts as expressed directly by the formulas and the underlying principles upon which their development was based.

Many excellent line graphs are used to illustrate the accumulation and discount concepts involved in the explanation of the principles being developed. Several pictorial graphs are also included that are subject to misinterpretation and as such have little or no value in a mathematics textbook.

The reviewer questions the desirability of the use of such names in the problem material as Hall and Prentice, Hill and McGraw, P and A Chain Stores, Kant Miss Fly-Swatter Company, Juke Boxes, Inc., and many others. Perhaps, however, they add a less serious touch to the material, and the students will find them stimulating.

The inclusion of tables of $S_{\overline{n}|i}^{-1}$ instead of the more usual $A_{\overline{n}|i}^{-1}$ would seem to be a matter of the personal choice of the author. Otherwise, the tables included are the usual, adequate tables required for compound interest and annuity concepts.

The solution of problems for interest rate and time concepts is presented by interpolation between appropriate tabular values rather than by means of logarithms. The topics of logarithms and the binomial theorem are presented in the appendix.

The reviewer feels that for a course between the usual arithmetic and mathematics of finance courses in content and rigor of presentation, the book merits favorable consideration by prospective users.—*Herbert Hannon, Western Michigan University, Kalamazoo, Michigan.*

Plane Geometry (with teacher's manual), John F. Schacht and Roderick C. McLennan (New York: Henry Holt and Company, 1957). Cloth, xvi+494 pp., $3.98.

Plane Geometry typifies in many respects the popular thinking on how this traditional course can best be presented to mathematics students in today's secondary schools. The question that is most pressing in the minds of an increasing number of mathematics teachers today, however, is, "Just how much longer must we continue to teach these somewhat antiquated, illogical Euclidean proofs and materials?"

The authors have made a conscious, visible effort to present the course in plane geometry as a logical system, building their theorems upon the orderly presentation of undefined terms, definitions, postulates, axioms, and constructions. The large number of theorems (ninety-two) is rather overwhelming. Perhaps it would have been better if some of these had been written in the form of original problems for the student.

The format of the book is generally pleasing. There has been a generous use of color, and there are many illustrations that have student appeal.

There is a thread of emphasis which runs throughout the book: the use of experimentation and deductive reasoning for the development of geometric concepts and relations. While few mathematics teachers will question the efficacy of this approach, many will undoubtedly feel that it is more applicable to units of intuitive geometry taught in grades six through nine.

This text presents an adequate supply of exercise materials and an unusually interesting number of items of enrichment in the form of historical notes, classical problems, and optional materials. The exercises have been graded to help the classroom teacher care for individual and class differences. Again, color has been used advantageously to indicate the various levels of difficulty and importance. A chapter summary, review, and test materials are to be found at the close of each chapter for the convenience and use of both teacher and pupil.

The authors have included units on spatial concepts, approximate numbers, and co-ordinate geometry. These are, however, placed in isolated "optional" units, thus giving them by implication the role of "unimportant supplementary materials." High school mathematics cannot be taught as a unified field as long as teachers and textbooks continue arbitrarily to pigeonhole those concepts that can provide many threads of continuity and unification.

This text has been well organized, and it presents an elementary course in plane geometry from a developmental point of view. Topics have been arranged in a natural order, and both inductive and deductive processes are used to advantage by the authors.

Even though the teaching of mathematics from the modern approach has gained considerable momentum in the past two years, we must recognize that traditional courses (particularly in geometry) will be taught in the public schools for years to come. Certainly this book deserves the careful examination of those teachers who will continue to teach the traditional Euclidean course in plane geometry.—*Ross A. Nielsen, Iowa State Teachers College, Cedar Falls, Iowa.*

Plane Geometry, A. M. Welchons, W. R. Krickenberger, and Helen R. Pearson (Boston: Ginn and Company, 1958). Cloth, viii+584 pp., $3.88.

In the judgment of this reviewer, this is an excellent traditional text. A great amount of material is presented. The teacher is advised as to what might be covered in a minimum course. Exercises are classified in A, B, and C lists in ascending order of difficulty. Some attention is given to three-dimensional geometry.

Color is used judiciously. Definitions, for example, are underlined in red. The result is a book of pleasing appearance that should be easy to use.

Many features of the text are to be commended. Logical concepts are discussed at regular intervals. Indeed, some of the logical assumptions that underlie mathematical proofs are specifically formulated and included in the list of postulates. Assumptions 36, 38, and 39 on pages 131 and 153 illustrate this.

The practical applications of geometry are emphasized. The treatment of regular polygons and circles is well done. Some notions of limits are presented in this chapter. Short chapters on elementary trigonometry and co-ordinate geometry are included.

The serious, able student who studies this text under a competent teacher will lay a substantial foundation for continuing mathematical growth. There is much talk today concerning revision of the high school mathematics curriculum. In particular, plane geometry is under heavy fire. This reviewer is convinced that change is inevitable in both algebra and geometry. But at the same time, any reasonable person must accord high respect to the type of mathematical instruction exemplified by the Welchons, Krickenberger, Pearson book. Texts like these, taught by competent teachers like these authors, have played an important role in the production of our present generation of mathematicians. They are not to be lightly discarded. It is to be granted that they tend to emphasize the *facts* of mathematics rather than its *structure*, but for many of the applications of mathematics, factual knowledge is of paramount importance. It is the prediction of this reviewer that in the long run we will find that we can well afford to devote a full year of the high school mathematics program to the study of geometry.

We shall make some criticism of this text, but it must be remembered that these criticisms apply generally to all traditional texts. As a matter of fact, if this were a review of any other traditional text, the criticisms would probably be sharper.

The treatment of logical concepts is not always clear. For example, on page 50, it is said that in the sentence, "An acute angle is an angle less than a right angle," the hypothesis is "An acute angle" and the conclusion is "is an angle less than a right angle." This example illustrates the concept of *definition* rather than *logical*

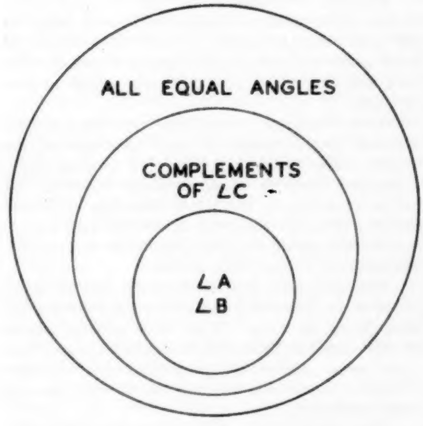


Figure 1

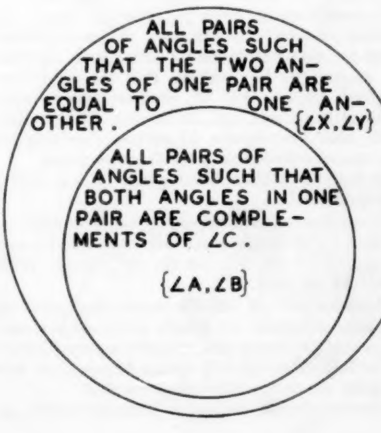
implication. In the light of the rich material presently available on logic, the sections on logical reasoning could be greatly improved.

Attention is drawn to the above circle diagram found on page 116 and presented to describe the deductive reasoning employed in inferring that *If* ∠A *and* ∠B *are complements of* ∠C *and complements of one angle are equal to one another, then* ∠A *and* ∠B *are equal to one another.*

The verbal argument runs: "Let a large circle contain all equal angles. . . ."

We ask: *What is meant by all equal angles? Equality is a relation*, not a *property*. The elements of the sets above should be *pairs* of angles, not individual angles. The diagram below correctly presents the ideas involved and probably represents what the authors actually have in mind.

Figure 2



The pair of angles $\{\angle A, \angle B\}$ is *one* element in the set of pairs of angles with each angle in one pair complementary to $\angle C$. The set of all these pairs is a subset of all pairs of angles such that the angles of one pair are equal to one another.

Over the past years there has been a progressive degeneration in the language of geometry. The teacher speaks of *a vertical angle, a parallel line, an equal segment.* It would be better to return to the purer language of Euclid and say instead: *one angle is vertical with respect to a second angle; one line is parallel to a second; one segment is equal to a second.*

Two geometry classes recently visited spent much time discussing the following concepts: *Is every point a vertex? What is a vertical angle?* It was obvious that the students thought that they were discussing *properties* rather than *relations.* Carefully written texts can prevent such confusion.

This text makes no clear-cut distinction between *Assumptions, Definitions,* and *Theorems.* It is implied that the assumptions of geometry fall into two categories—axioms and postulates—and that these two sets are essentially distinct in nature. This has been known for sixty years as a logically untenable point of view.

Some fifty-seven assumptions are presented. Of these, approximately twelve are needed, although several of these twelve are stated in such a way as to make them logically useless. For example, the addition postulate *If equals be added to equals, the sums are equal to one another* is presented as if it is a postulate concerning number properties. What is needed here are three distinct postulates that deal separately with *addition of segments, addition of angles,* and *addition of polygons.* No postulate is necessary for addition of *numbers.* It is to the credit of the authors that they do present an intuitive discussion of addition of segments and angles.

The forty-odd assumptions other than the few mentioned above fall into three categories. Each of these remaining assumptions is an easy theorem, a definition, or a useless statement.

Approximately twenty-seven of these remaining assumptions are easy theorems, some fifteen should be replaced by definitions, and the rest appear to be useless. For example, Assumption 11, on page 56, asserts: *One straight line and only one can be drawn through two points,* and Assumption 12 states: *Two straight lines cannot intersect in more than one point.*

It is obvious that Assumption 12 is a trivial consequence of 11.

Examples of assumptions that should be replaced by definitions are Assumptions 24 and 25 (p. 56), 26 (p. 57), 34 (p. 98), 35 (p. 129), 41 and 42 (p. 288).

Assumption 34 is: *Corresponding parts of congruent polygons are equal; polygons are congruent if their corresponding parts are equal.* This is a definition of what it means to say that two polygons are congruent to one another.

Assumption 42 states: *A point is within, on, or outside a circle if its distance from the center is less than, equal to, or greater than the radius; and conversely.* What we have here is actually a definition of three concepts, namely, *circle, inside of a circle,* and *outside of a circle.*

Examples of useless or meaningless assumptions are Assumptions 6 and 8 on page 53. Assumption 6 asserts that *a quantity may be substituted for its equal.* This is vague and certainly not needed. A student may of course use Assumption 6 to argue that if $AB = CD = EF$, then $AB = EF$, but he may also use the transitivity property of *equality of segments.* Indeed, consider the statement, "AB, BC, and CA are sides of a triangle." Now let $A'B'$ be *equal* to AB and substitute $A'B'$ for AB in the above statement. The resulting statement is *not* necessarily equivalent to the original.

We summarize our judgment of this book by remarking that the students who study geometry from this text should learn many of the facts of geometry and develop some skill in proof, but they will gain very little insight into the *logical principles* upon which mathematical proofs are based. They will have no clear understanding of the *structure* of geometry as a mathematical system. They probably will not be able to distinguish between *defined terms* and *undefined terms.* Nor will they be able to differentiate between *assumptions, definitions,* and *theorems.*—*Charles Brumfiel, Ball State Teachers College, Muncie, Indiana.*

Plane Geometry for Colleges, L. J. Adams (New York: Henry Holt and Company, 1958). Cloth, x+214 pp., $3.50.

According to the author, this textbook is designed "for use in a one-semester course in plane geometry for college students. It can be used to prepare the student for further work in mathematics or to serve as a terminal course in the subject." Nothing is indicated as to whether it is assumed that the student has never studied geometry in high school or whether this is to be thought of as a "refresher course." (To be sure, one might be forgiven for asking why any student should be admitted to a college without having studied plane geometry in high school, but it is being done.) In any event, the book under consideration is likely to prove somewhat disappointing, chiefly because of its regrettable brevity and oversimplification.

Succinctness is highly desirable, especially in a textbook on geometry. Many high school texts err in the opposite direction, running to five hundred pages or more, with more than an ample supply of original exercises. The present volume, however, is so brief and simple that one wonders how far a book should go in "talking down" to the reader. It may be assumed that the college freshman is at least considerably more mature than he was when in the tenth grade. It is therefore reasonable to expect that, on the college level, the student of elementary plane geometry might really achieve what we have always hoped the high school pupil would gain from geometry,

namely, significant understanding of the nature of deductive reasoning; the distinction between inductive and deductive thinking; the nature of assumption, truth, and implication; the relationship between a theorem, its converse, inverse, and contrapositive; logical equivalence; types of indirect proof, and so on.

To be sure, these matters are touched upon, but not at all with the emphasis which they deserve. For example, only a little more than half a page is devoted to the discussion of converse, inverse, and contrapositive; less than a page is devoted to a discussion of indirect proof; the words "implication," "inductive," and "syllogism" do not appear in the index; there are no allusions to Euler circles or Venn diagrams; the existence of non-Euclidean geometries is dismissed with two sentences (page 43).

We note that the classical distinction between "axioms" and "postulates" has been retained, which, while not a grievous sin, is at variance with general practice. We also find the admonition that "since so much of geometry is concerned with proving theorems, it is essential that the student attempt to develop a *scientifically critical attitude* [italics ours]. This attitude could very well carry over into other subjects and daily life situations."

This seems to us a bit fuzzy: if a "scientifically critical attitude" implies hypothetico-deductive reasoning, logical rigor, and axiomatics, the term would seem to be ill-chosen; if it implies informal, experimental, and intuitive procedures of discovery, there is little or none of this in the book. And as for carrying over deductive reasoning "into other subjects and daily life situations," that, too, is conspicuous for its absence or paucity. The nearest approach to "exercises in reasoning" are three rather ill-advised problems (pages 70–71) of the "conductor, engineer, fireman, Brown, Jones, Smith, but not respectively" type. Interesting enough per se, they have but little connection with deductive geometry or logical reasoning as used in ordinary discourse.

The book is well organized, being divided into nine chapters. These discuss, after the introduction, straight line figures; parallels; circles; loci and concurrent lines; similar polygons; areas; regular polygons; applications. The diminished emphasis on "construction problems" is welcome, although we miss any substantial treatment of inequalities.

A number of other things can also be said in the book's favor. The typography, general format, and diagrams are refreshingly clear and attractive; the propositions selected for discussion and proof represent a commendable choice; the text and exposition are unusually crisp and readable.

We have the uncomfortable feeling, however, that the book is admirably suitable for use in the eighth grade. If we are not altogether wrong in this judgment, this may be an unwitting commentary on the deplorable status of the general run of contemporary college freshman offerings. —*W. L. Schaaf, Brooklyn College, Brooklyn, New York.*

The Mathematics Student Journal

The solutions below are to problems appearing in *The Mathematics Student Journal,* Vol. VI, No. 1.

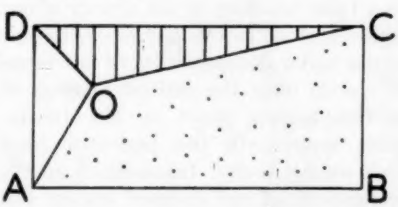
Problem 123. Two black squares have been cut off the checkerboard. Thus 30 black and 32 white squares remain. But each card covers one white and one black square. After placing 30 cards on the board, 2 white squares remain. Since they are both white they cannot be neighbors and cannot be covered by a single card.

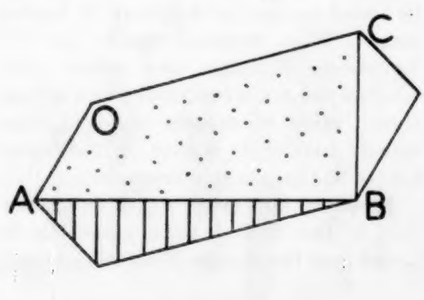
Problem 124. It is possible to cement opposite sides of a rubber hexagon together, and the resulting shape is again a bicycle inner tube. The simplest way to see this is perhaps the following. Suppose an inner tube is made by cementing together opposite sides of the rectangle *ABCD* shown here. Now let cuts be made along the lines *AO, CO, DO.* Open out: a hexagon results.

Please do not tell students the above solution too quickly. They should have anywhere from a week to a month to ponder the problem. One might advise students to experiment with actual material—a rubber sheet, cloth, plastic, knitted material, or suchlike.

Problem 125. Solution not published here. Students should send their work to Oscar Schaaf, University of Oregon.

Solutions to the problems mentioned by Professor Ross of Notre Dame University can be obtained by sending a stamped, addressed envelope to him.





Edited by Joseph N. Payne, University of Michigan, Ann Arbor, Michigan, and William C. Lowry, University of Virginia, Charlottesville, Virginia

Teaching digit problems in algebra

by Dorothy Fish, Pandora-Gilboa High School, Pandora, Ohio

One of the important concepts in arithmetic is place value in our number system. Many students reach high school without a complete understanding of this basic characteristic of our system of numeration. That their understanding of place value is incomplete is shown when pupils have unusual difficulty with digit problems in algebra. To understand digit problems and their solution, the student must have a clear understanding of place value.

Before introducing digit problems, I give a brief summary of the history of our number system. In the discussion, I point out the major developments of our number system from the earliest tracings of the Babylonians down to the Hindu-Arabic system, in the perfected form which we use today. Information on the development of our decimal system can be found in any good history of mathematics. When students realize that the Egyptians, Romans, and other early scholars did not recognize the idea of positional value to express numbers, they usually have more respect for the inventors of our system of numeration.

Probably no better visual representation of the idea of place value can be found than the abacus. A model can easily be made with a little wire, a small picture frame, and a bag of kindergarten beads. Freshman students enjoy using the abacus and it helps them understand the value represented by the digits in different positions. For example, if 235 is represented on the abacus, it is quite clear that the digits 2, 3, and 5 represent two hundreds, three tens, and five units, respectively.

When the idea of place value is understood, the class is ready for digit problems. I begin by using the abacus to illustrate the change in value of a number by the reversal of its digits. After a few examples, students find it easy to write the equations for the digit problem. After solving the equations, I get some student to use the abacus to illustrate the solution to the class.

I find the results more beneficial if there are several abacus models (possibly from the first-grade teacher) available so that different students can solve a problem at the same time. From my own experience the amount of effort and time formerly required to explain digit problems in elementary algebra has been greatly reduced. In addition, the students have a much richer appreciation of place value, and they have learned some interesting facts about the history of mathematics.

• NOTES FROM THE WASHINGTON OFFICE

Edited by M. H. Ahrendt, Executive Secretary, NCTM, Washington, D.C.

Annual financial report

Attached is a brief financial report of the Council for the fiscal year ending May 31, 1958. When the budget for the year was made, all indications were that there would be a substantial financial loss. Two circumstances intervened to produce a small financial gain. The membership increased by 20 per cent during the year, sharply increasing the receipts from memberships and subscriptions. The *Twenty-third Yearbook, Insights into Modern Mathematics,* enjoyed a vigorous sale, bringing in an unexpectedly large income. This gain, coupled with the new dues, should give the Council financial stability for many future years and should make possible the implementing of desirable but previously impossible projects.

Receipts and expenditures of The National Council of Teachers of Mathematics for the fiscal year, June 1, 1957-May 31, 1958

June 1, 1957—Total cash resources		$ 42,569.20
RECEIPTS		
Memberships with THE MATHEMATICS TEACHER subscriptions	$ 38,388.50	
Memberships with *The Arithmetic Teacher* subscriptions	13,828.23	
Institutional subscriptions to THE MATHEMATICS TEACHER	16,164.27	
Institutional subscriptions to *The Arithmetic Teacher*	10,808.89	
Subscriptions to *The Mathematics Student Journal*	4,366.55	
Sale of advertising space in THE MATHEMATICS TEACHER	6,268.90	
Sale of advertising space in *The Arithmetic Teacher*	1,741.67	
Interest on U. S. Treasury Bonds	375.00	
Net profit from conventions	301.90	
Miscellaneous	92.29	
Sale of publications		
Yearbooks	28,120.27	
Supplementary publications, etc.	13,088.01	
Total receipts	$133,544.48	
EXPENDITURES		
Washington office	$ 44,512.19	
President's office	2,375.03	
Vice-Presidents' office expenses	135.00	
THE MATHEMATICS TEACHER	32,248.55	
The Arithmetic Teacher	12,242.18	
The Mathematics Student Journal	3,022.93	
Committee work	3,637.03	
Travel by Board members	3,133.42	
Preparation and printing of yearbooks	11,436.84	
Preparation and printing of supplementary publications	8,077.34	
Distributing "As We See It"	1,255.08	
Storage and shipment of publications, miscellaneous	693.01	
Total expenditures	$122,768.60	
Increase in cash resources		$10,775.88
May 31, 1958—Total cash resources		$53,345.08

NCTM

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Minutes of the Annual Business Meeting

Hotel Cleveland, Cleveland, Ohio

April 11, 1958

Dr. Howard F. Fehr, President, called the meeting to order at 4:10 P.M.

I. A motion was made, seconded, and passed to approve the minutes of the meeting of March 29, 1957, as printed in the *Journals*.

II. In his opening remarks, Dr. Fehr expressed his appreciation to his fellow officers, the Board of Directors, and the membership in general for the fine co-operation he had received during his administration. He stated further that the Council was continuing to make fine progress in meeting its objectives, this progress being due to the excellent work of the members. In addition, Dr. Fehr made the following announcements:

A. The first printing of the Twenty-third Yearbook (5,000 copies) has been sold and a new printing authorized. This yearbook, *Insights into Modern Mathematics*, was under the editorship of Dr. F. L. Wren.

B. No yearbook is being published during 1958; however, three yearbooks will be published in successive years beginning with 1959.

C. A new editor of *The Mathematics Student Journal* has been appointed to replace Dr. Max Beberman whose term expired. The new editor is Mr. W. W. Sawyer of Wesleyan College, Connecticut. A new format is being designed for the *Journal* which will enhance its attractiveness and make it easier to use and file. Dr. Beberman was complimented for the work which he did during his term as editor.

D. The new Guidance Pamphlet has been completed and will soon be available. (See Part A. of the President's report, page 464 of THE MATHEMATICS TEACHER [October, 1957] and page 229 of *The Arithmetic Teacher* [November, 1957].) The title will be *Mathematics People and a Good Life*. The National Science Foundation is sponsoring the project and will print at least one million copies, which will be freely distributed.

E. A special committee composed of Dr Philip Peak, Dr. Henry Van Engen, Dr F. L. Wren, and Dr. Robert E. Pingry Chairman, has been appointed to prepare a pamphlet on "The Role of The National Council of Teachers of Mathematics in Mathematics Education." The first draft has just been presented to the Board. After a few changes this pamphlet will be published (by June 1) and distributed.

F. The Board invited Dr. Daniel B. Lloyd to prepare a manuscript on the history of the Council. The manuscript was presented at this session of the Board and has been referred to the Publications Board.

III. Dr. Houston T. Karnes, Recording Secretary, gave a brief review of the actions of the Board during the past year. This review included items which were thought to be of interest to the members. The review follows:

A. Last year it was announced that the dues, in all probability, would be raised. This was done at the 1957 Summer Meeting. The new rates, effective October 1, 1958, are as follows:

1. The individual membership dues will be $5.00. This includes one Journal. Both Journals may be had for $8.00.
2. The student membership dues shall remain the same: $1.50 for one Journal and $2.50 for both Journals.
3. The institutional subscription rates shall be $7.00 for each Journal or both Journals for $14.00

B. The new president will be the official representative of the Council at the International Congress in Edinburgh, Scotland, this summer. In addition, two other members will represent the Council. They are Dr. Howard F. Fehr and Dr. Bruce Meserve.

C. An Addressograph system is being installed in the Central Office. This will

increase the efficiency of the office to the members and to related organizations.

D. The 1958–59 budget will not be presented here since it is published elsewhere in this issue. It is well to point out, however, that the general operation budget is $124,415.00. This figure represents an increase of some $23,000 over last year due to increased costs and an increase in membership. For the first time in several years, the budget is balanced without resorting to the Publications and Reserve Account.

E. Future Meetings:
   1. Summer, 1958, Greeley, Colorado. An excellent program has just been completed under the direction of Vice-Chairman Alice M. Hach.
   2. Christmas, 1958, New York City. The dates are December 29 and 30. This will give those in attendance an opportunity to take part in the New Year festivities around Times Square. An excellent program is being prepared by Vice-Chairman Robert E. Pingry.
   3. Spring, 1959, Dallas, Texas.
   4. Summer, 1959, University of Michigan.
   5. Spring, 1960, Buffalo, New York
   6. Summer, 1960, Salt Lake City, Utah.
   7. Spring, 1961, Chicago, Illinois.

F. Since financial support for the work of the Secondary School Curriculum Committee has not been found, the Board agreed to advance up to $10,000 so that the Committee could begin to work on the program which has been devised. This amount will not go far, but it will enable the Committee to produce, within a year, some of the most needed aspects of its program: particularly that phase dealing with grades seven, eight, and nine. There is still hope that the necessary money will be found to complete the entire study.

G. Yearbooks:
   1. The Twenty-fourth Yearbook, under the editorship of Dr. Phillip S. Jones, is due to be published in 1959. The title is *Central Themes and Concepts*.
   2. The Twenty-fifth Yearbook, under the editorship of Dr. F. E. Grossnickle, is due to be published in 1960. The title is *Arithmetic*.
   3. The Twenty-sixth Yearbook, under the editorship of Dr. Donovan A. Johnson, is due to be published in 1961. The title is *Evaluation*.

H. Dr. Myron F. Rosskopf has been appointed editor to plan and edit the March, 1959, issue of *The Bulletin* of the National Association of Secondary School Principals. This is by way of an invitation from the Association. It is the second invitation for such participation on the part of the Council. This issue will cover significant movements in Mathematics Education.

IV. Mr. M. H. Ahrendt, Executive Secretary, gave the following brief report:

A. As of April 1, 1958, the total membership of the Council had reached 19,500. This is a major increase over last year. Miss Mary C. Rogers and the Membership Committee were complimented for the fine work accomplished.

B. As of 4:00 P.M. (today) the total registration for this meeting was 1,940. This is the largest meeting we have had.

C. Due to the growth and increased activity of the Council, the office staff has grown from two persons in 1951 to nine members in 1958.

D. The Council is presently housed in an office space of 800 square feet in the new NEA Building. Since these quarters are already cramped, office space of 1200 square feet is promised for the coming fall.

E. The Secretary pointed out that in an operation of the Council's proportions errors were bound to occur, especially in the membership lists. He invited all to inform him whenever errors occur.

F. Since the fiscal year of the Council does not end until May 31, no attempt was made to give a complete financial report. Such a report will be published in a fall issue of the two Journals. The Secretary did indicate, however, that general operations would not require as much from the reserve account as was anticipated when the current budget was approved.

G. Five new pamphlets were added to the list of Supplementary Publications during the year.

H. The Secretary stated that the Editorial Staff of the N.E.A. had been of great help to him during the past year.

I. In closing, the Secretary paid his compliments to the outstanding work of the State Representatives during the year.

V. Dr. Clifford Bell, Chairman of the Nominating and Election Committee, gave the following report of the recent balloting:

President: Dr. Harold P. Fawcett
Vice-President for Senior High School: Miss Ida May Bernhard
Vice-President for Elementary School: Dr. E. Glenadine Gibb
Directors: Mr. Frank B. Allen
Dr. Burton W. Jones
Dr. Bruce E. Meserve.

VI. Dr. Fehr introduced the new officers and Directors as declared elected by Dr. Bell.

VII. Dr. Milton W. Beckmann, Chairman of the 1959 Committee on Nominations and Elections, invited the members to send in recommendations. The vacancies to be filled are: Vice-President for College, Vice-President for Junior High School, and three Directors.

VIII. Under the heading of New Business there was a discussion concerning an approach toward the uniformity of mathematical symbolism. Although there was no motion, several suggested that the Board take this matter under advisement.

IX. In his closing remarks, Dr. Fehr again expressed his appreciation to the officers, directors, editors, committee chairmen, committee members, and the membership at large for the wonderful co-operation and help which he had received during his administration.

X. Report of the Resolutions Committee, composed of Dr. Phillip S. Jones and Dr. H. Vernon Price, Chairman. Dr. Price gave the following report:

> WHEREAS the local committees listed in your convention program have assumed most of the responsibility for arrangements with hotels, schools, and industry, and
>
> WHEREAS the magnitude of these tasks and the cheerful efficiency with which they have been executed are obvious to all of us in attendance at this the largest meeting in the history of the Council,
>
> THEREFORE, be it resolved that The National Council of Teachers of Mathematics acknowledge in this open and public meeting its indebtedness and appreciation to the teachers of the Cleveland area and to the Greater Cleveland Mathematics Club in particular for their efforts in making this convention a memorable success.

A motion was made and seconded to adopt the resolution presented by the Committee. The motion was passed unanimously by a standing ovation.

XI. There being no further business, the meeting was duly adjourned at 4:50 P.M.

Respectfully submitted,
Houston T. Karnes,
Recording Secretary

The 1958 budget

The Budget Committee for 1958 was composed of Miss Agnes Hebert, Mr. Jackson B. Adkins, and Dr. Houston T. Karnes, Chairman. Under the directive of President Howard F. Fehr, the committee met at the Central Office in Washington during the week end of January 25, 1958. It was President Fehr's wishes that the committee work out a proposed budget in the meeting at the Central Office and circulate the proposed budget to the members of the Board at least one month prior to the annual meeting. In view of the importance of the budget, the committee feels that the method followed this year is most desirable. The committee had the fine services of Executive Secretary M.H. Ahrendt, and the availability of all the important records in the Central Office. The budget was discussed at the Annual meeting of the Board on April 8, at Cleveland, Ohio. The budget as adopted by the Board is submitted below. Few changes in the report were made by the Budget Committee. The adopted budget is predicated upon the new dues as of October 1, 1958.

RECEIPTS

Memberships and subscriptions	$113,040	
MSJ Subscriptions	5,000	
Advertising in journals	6,000	
Interest on U.S. Treasury bonds	375	
Total	$124,415	$124,415

Expenditures		
Washington Office		
Executive Secretary		
Salary	$ 10,000	
Special Benefits	1,100	
Travel	1,000	
Secretarial and clerical help		
Salaries	30,000	
Special benefits	1,000	
Office expenses	11,000	
Membership work	500	
Total	$ 54,600	
President's Office		
Office expense	$ 1,500	
Travel	1,000	
Special Projects Fund	700	
Total	$ 3,200	
Vice-Presidents		
Postage and secretarial help ($100 each)	400	
Total	$ 400	
The Mathematics Teacher		
Printing and mailing	$ 27,000	
Editorial and office expense	4,500	
Travel, Editor	500	
Total	$ 32,000	
The Arithmetic Teacher		
Printing and mailing	$ 13,000	
Editorial and office expense	1,500	
Travel, Editor	500	
Total	$ 15,000	
The Mathematics Student Journal		
Printing and mailing	$ 3,000	
Editorial and office expense	1,000	
Travel, Editor	250	
Total	$ 4,250	
Other		
Affiliated Groups (travel of chairman to Annual Meeting and for *Newsletter*)	$ 500	
Committee work	2,500	
Travel, Directors	4,000	
Total	$ 7,000	
Reserve	7,965	
Total	$ 7,965	
Total Expenditures		$124,415

Minutes of the Ninth Delegate Assembly

Hotel Cleveland, Cleveland, Ohio

April 9–12, 1958

Edited by Elizabeth Roudebush, Seattle Public Schools, Seattle, Washington

The Ninth Delegate Assembly for Affiliated Groups of The National Council of Teachers of Mathematics met in the Rose Room of the Hotel Cleveland, Cleveland, Ohio, at 9:00 A.M., April 10, 1958, with Chairman Elizabeth J. Roudebush presiding.

Dr. Howard F. Fehr, President of The National Council of Teachers of Mathematics, spoke to the group, emphasizing the National Council's position in providing strong leadership in developing the modern mathematics curriculum. He pointed out the change in emphasis from rote learning to concept building, stressing that mathematics courses will contain some new mathematics. The most important change will be a new point of view toward the subject itself. The mathematicians, the mathematics educators, and the secondary school teachers have a definite responsibility in making mathematics a more meaningful subject.

The delegates and alternates were introduced by their respective regional representatives.

How we publish our newsletter

A panel of four newsletter editors discussed "How We Publish Our Newsletter." Mrs. Adeline Riefling, St. Louis, Missouri, panel moderator, reviewed the main purposes and methods of producing a newsletter. She shared her experiences as editor of the Missouri Newsletter, indicating the problems of collecting material and reproducing it.

Clarence H. Heinke, Capital University, Columbus, Ohio, editor of the Ohio Newsletter, discussed the content and publication of two issues per year. Their newsletter reviews important papers presented at mathematics conferences and announces programs for the coming meetings. The Ohio Council of Teachers of Mathematics prints about 1200 copies at a cost of from $65 to $105 per issue and uses some copies of the newsletter to aid in getting new members.

Gertrude Hendrix, University of Illinois, Urbana, Illinois, discussed problems involved in offset printing of 800–850 copies of the Illinois Newsletter. The cost of the four issues ranges from $75 to $120 per issue, depending on size. They are experimenting, at present, in selling advertising space to help cover the cost of publishing.

Madeline Messner, Abraham Clark High School, Roselle, New Jersey, gave a brief résumé of the fourteen years of publishing the New Jersey Newsletter. They have three issues, with 1500 copies costing approximately $250 per issue. Commercial advertising covers the cost of one issue. They have an editorial board with each level of teaching represented.

C. E. Flanagan, Wisconsin State College, Whitewater, Wisconsin, reported that the Wisconsin Newsletter is published three times a year. He is assisted by an editorial committee of three. The secretary of the Wisconsin Mathematics Council prepares the gummed address labels. They print about 200 copies at a cost of $50 per issue.

Uses of mathematical contests

John C. Bryan, North High School, Omaha, Nebraska introduced L. G. Yoder, East Palestine, Ohio. Mr. Yoder, of the Yoder Instrument Company, described a local, project-type of mathematical contest carried on at East Palestine. He said that we must develop better scientists and engineers if we are to survive and that mathematics teaching plays a very important part in this problem. Prizes are used to stimulate interest. Good newspaper publicity for the contest serves as a means of making mathematics more important to both students and citizens of the community.

Daniel B. Lloyd, District of Columbia Teachers College, Washington, D. C., discussed the 80-minute, objective-type test prepared under the auspices of the Mathematics Association of America, which is used to determine outstanding mathematics students in high schools participating in the contest. Approximately 85,000 students competed in this contest in 1958. He emphasized the danger in carrying any type of contest to extremes.

Committee reports

Curriculum. Frank B. Allen, Lyons Township High School and Junior College, La Grange, Illinois, reported that the Secondary School Curriculum Committee has organized its work into ten specific areas of curriculum study. He emphasized the need for financial assistance to carry on the work of these subcommittees. The help and encouragement of the membership is

vital to the progress of the committee at this time.

Television. Joe Hooten, Florida State University, Tallahassee, Florida, told of some of the obstacles that the Television Committee is facing. Although the work of the committee is in its initial stage, the following recommendations were made:

1. That a subcommittee be appointed to experiment and study the effectiveness of teaching by television.
2. That a study be made of the effectiveness of television as a promotional agency for mathematics.
3. That space be made available for use of this committee in the National Council journals.
4. That financial help be solicited, subject to approval of the National Council.

Supplementary Publications. Henry Swain, New Trier Township High School, Winnetka, Illinois, told the group that this committee had charge of all publications of The National Council of Teachers of Mathematics except the yearbooks and the journals. The committee solicits and edits articles for publication. He asked the help of the Delegate Assembly in securing manuscripts, listing timely topics, and suggesting people to write for future publications.

Membership. Mary C. Rogers, Roosevelt Junior High School, Westfield, New Jersey, gave a very enthusiastic report concerning the membership growth and the future goals of The National Council of Teachers of Mathematics. The membership February 1, 1958 was 18,324, with an estimated membership of 19,500 today. Members were urged to work for continued growth in the organization.

Report from Puerto Rico. Francisco Garriga, University of Puerto Rico, discussed the organization of a Council of Teachers of Mathematics in Puerto Rico. He expressed appreciation to Dr. Howard F. Fehr for his help during the time he spent in Puerto Rico. They have applied for affiliation with The National Council of Teachers of Mathematics.

Dues. M. H. Ahrendt, Executive Secretary of The National Council of Teachers of Mathematics, explained the new schedule of dues and the need for the schedule.

Invitation from Dallas. A. W. Harris issued a cordial invitation to the members of the National Council to the Dallas meeting April 1–4, 1959.

Miss Roudebush expressed her appreciation to all who participated in the meeting. The meeting of the Ninth Delegate Assembly adjourned at noon.

Respectfully submitted,
Grace Arbogast,
Mildred B. Cole,
Secretaries of Delegate Assembly

The following people were chosen by their local groups as delegates and alternates to the Delegate Assembly. Where two names are listed, the first is the delegate, and the second one the alternate.

Arkansas Council of Teachers of Mathematics
Miss Dorothy Long, Arkansas Teachers College, Conway, Arkansas

California Mathematics Council
Marian C. Cliffe, 1457 Winchester Avenue, Los Angeles 1, California
John D. Hancock, Capuchino High School, San Bruno, California
John Forster, 351 North Highland Place, Alhambra, California

Colorado Council of Teachers of Mathematics
Forest Fisch, Colorado State College, Greeley, Colorado

Delaware Council of Teachers of Mathematics
Russell Dineen, Wilmington High School, Wilmington, Delaware

District of Columbia Teachers of Mathematics
Daniel Lloyd, D. C. Teachers College, Washington, D. C.
Miss Jane Hill, 3051 Harrison Street N.W., Washington 12, D. C.

Benjamin Bannekar Mathematics Club
Mrs. Juanita S. Tolson, 723 Jefferson Street, N.W., Washington, D. C.

Florida Council of Teachers of Mathematics
J. Richard Sewell, 1017 Aloma Avenue, Winter Park, Florida

Dade County Council of Teachers of Mathematics
Michael Kambour, 2240 S. W. 80th St., Miami, Florida

Pinellas County Council of Teachers of Mathematics
Mrs. Carol S. Scott, 1190 8th Street North, St. Petersburg, Florida
Miss Ruth T. Reynolds, 701 Ft. Harrison Avenue, Clearwater, Florida

Georgia Council of Teachers of Mathematics
Miss Amabel Landsell, Tukman Junior High School, Augusta, Georgia

Illinois Council of Teachers of Mathematics
LeRoy Sachs, Cahokia-Commonfields High School, Range & Jerome Lane, East St. Louis, Illinois
Miss Ruby Cooper, 1218 Jersey, Quincy, Illinois

Chicago Elementary Teachers Mathematics Club
Joseph J. Urbancek, 1112 Grant, Evanston, Illinois

Men's Mathematics Club of Chicago and Metropolitan Area
Bro. Norbert, C.S.C., Holy Trinity High School, 1443 W. Division Street, Chicago, Illinois

Women's Mathematics Club of Chicago and Vicinity
Miss Maude Bryan, 3108 Haussen Court, Chicago 18, Illinois

Indiana Council of Teachers of Mathematics
Mrs. Eleanor Guyer, Southport High School, Southport, Indiana

Gary Council of Teachers of Mathematics
Harold R. Jones, 4617 Delaware Street, Gary, Indiana
Miss Olive Leskow, 234 West 49th Avenue, Gary, Indiana
Iowa Association of Mathematics Teachers
Dr. Ross Nielsen, Laboratory School, I.S.T.C. Cedar Falls, Iowa
Kansas Association of Teachers of Mathematics
Miss Gertrude Welch, Shawnee-Mission High School, Merriam, Kansas
Wichita Mathematics Association
Dr. Lottchen Hunter, 1706 N. Market, Wichita 4, Kansas
Louisiana-Mississippi Branch of the National Council of Teachers of Mathematics
Mrs. A. H. Wehe, 6202 Goodwood Avenue, Baton Rouge, Louisiana
Mathematics Section of the Maryland State Teachers Association
Mrs. Margaret A. Bowers, Marriottsville, Maryland
Alfred E. Culley, 3710 Lochearn Drive, Baltimore 7, Maryland
Mathematics Teachers of Prince George's County
Gordon A. Patterson, Surrattsville Junior-Senior High School, Clinton, Maryland
Michigan Council of Teachers of Mathematics
Miss Catherine Meehan, 2772 Military, Port Huron, Michigan
Detroit Mathematics Club
Sigfrid Anderson, 14845 Mark Twain, Detroit
Missouri Council of Teachers of Mathematics
Mrs. Adeline A. Riefling, 3507 Hawthorne Boulevard, St. Louis 4, Missouri
Nebraska Section—National Council of Teachers of Mathematics
John Bryan, Omaha North High School, Omaha, Nebraska
David Wells, University of Nebraska, Lincoln, Nebraska
The Association of Teachers of Mathematics in New England
Jackson B. Adkins, Box 49, Exeter, New Hampshire
Prof. Henry W. Syer, Boston University School of Education, 332 Bay State Road, Boston
Association of Mathematics Teachers of New Jersey
Mrs. Lina Walter, State Teachers College, Paterson, New Jersey
Max A. Sobel, State Teachers College, Montclair, New Jersey
Association of Teachers of Mathematics of New York State
Mrs. Dorothy Lape, 315 Washington Highway, Buffalo, New York
Mrs. Miriam Howard, Phelps Central School Phelps, New York
Association of Teachers of Mathematics of New York City
Dr. Samuel Greitzer, Bronx High School of Science, 120 E. 184th Street, Bronx 53, New York
Nassau County Mathematics Teachers Association
Mrs. Florence Elder, West Hempstead Junior-Senior High School, West Hempstead, New York
Ohio Council of Teachers of Mathematics
Miss Helen Brown, Steubenville High School, Steubenville, Ohio
Mathematics Club of Greater Cincinnati
Mrs. Helen S. Walter, 522 Overhill Lane, Cincinnati 38, Ohio
Greater Cleveland Mathematics Club
Mrs. Helen J. Scheu, 7802 Wainstead Drive, Parma, Ohio
Richard Keuchle, 3573 Normandy Road, Shaker Heights 22, Ohio
Greater Toledo Council of Teachers of Mathematics
Joseph Jordan, Maumee High School, Maumee, Ohio
Oklahoma Council of Teachers of Mathematics
Coy Pruitt, Tulsa, Oklahoma
Oklahoma City Mathematics Teachers Council
Miss Norma Louise Jones, 1508 S. W. 43rd., Oklahoma City 19, Oklahoma
Oregon Council of Teachers of Mathematics
Mrs. Lyle Mary Wheeler, Astoria Public Schools, Astoria, Oregon
Pennsylvania Council of Teachers of Mathematics
Dr. Catherina A. V. Lyons, 12 S. Fremont Avenue, Pittsburgh 2, Pennsylvania
Mathematics Council of Western Pennsylvania
Earle F. Myers, University of Pittsburgh, Pittsburgh, Pennsylvania
John C. Knipp, University of Pittsburgh, Pittsburgh, Pennsylvania
Association of Teachers of Mathematics of Philadelphia and Vicinity
Miss Kathryn Clark, Girls High School, Philadelphia, Pennsylvania
Texas Council of Teachers of Mathematics
Arthur W. Harris, 4701 Cole Avenue, Dallas
Greater Dallas Mathematics Association
Arthur W. Harris, 4701 Cole Avenue, Dallas
Houston Council of Teachers of Mathematics
Mrs. Thelma Hammerling, 1502 Fairview, Houston 6, Texas
Utah Council of Teachers of Mathematics
Miss Eva Crangle, West High School, Salt Lake City, Utah
Mathematics Section, Virginia Education Association
Colonel William Mack, Douglas Freeman High School, Richmond, Virginia
Arlington County Council of Teachers of Mathematics
Mrs. Mary D. Campbell, 608 North Lincoln Street, Arlington 1, Virginia
Miss Alice R. Bolton, 1101 North Kenilworth Street, Arlington 5, Virginia
Washington State Council of Teachers of Mathematics
Robert Willson, Eisenhower High School, Yakima, Washington
Puget Sound Council of Teachers of Mathematics
Miss Elizabeth Roudebush, 815 4th Avenue North, Seattle 9, Washington
Wisconsin Mathematics Council
Prof. C. E. Flanagan, Whitewater State College, Whitewater, Wisconsin

Nineteenth Christmas Meeting

Hotel Sheraton-McAlpin, New York City

December 29, 30, 1958

The Christmas meeting of the Council will be held in the Hotel Sheraton-McAlpin, Broadway at 34th Street in New York City. Outstanding leaders in the field of mathematics and mathematics education will appear on the program and will discuss important and timely topics.

Dr. James B. Conant, former President of Harvard University and United States High Commissioner for Germany, will be the speaker at the general session on Monday evening, December 29. Dr. Conant has recently been engaged in a study of the American high school. He will speak on the subject, "The Place of Mathematics in the Comprehensive High School."

At the banquet on Tuesday evening, Dr. C. V. Newsom, President of New York University, will give the address. Dr. Newsom will speak on a timely topic on mathematics education in America.

Dr. Albert Meder, Jr., will give the opening address on Monday morning on the topic, "Sets, Sinners, and Salvation." Dr. Meder has recently completed his work as Executive Director of the Commission on Mathematics of the College Entrance Examination Board. Since this commission is publishing a report during the last part of 1958, Dr. Meder's remarks will be of special interest.

Because of the tremendous interest of mathematics teachers and others in the problems of revising the high school mathematics curriculum, one general session and several special sections will be directly on these problems. On Tuesday morning, December 30, Frank Allen, who is chairman of the Council's Committee on the Secondary School Curriculum, will be the speaker. He will discuss the subject, "The Teacher's Role in Curriculum Development." Following this address, Professors Max Beberman, Robert E. K. Rourke, Warwick W. Sawyer, and Robert Davis will hold a panel discussion-debate on the topic, "The High School Curriculum in Mathematics Ten Years from Now." At this session several issues relating to aims and methods of mathematics instruction will be discussed.

Many schools are also planning on the separation of their students into special classes for special instruction in mathematics. This operation requires that careful selections be made. Sheldon Myers, of the Educational Testing Service, is going to discuss "Problems and Procedures for Selection of Students for Various Mathematics Programs."

College teachers will be interested in hearing Dr. Albert A. Bennett present his ideas on "College Freshman Mathematics Courses," and Dr. William L. Duren discuss "What Changes in Mathematics Curriculum Should Colleges Be Making?"

Elementary teachers can hear Dr. M. L. Hartung, Dr. Vincent Glennon, Dr. Nathan Lazar, and others. Papers will be presented on "Distinguishing Between Basic and Superficial Ideas in Arithmetic," "Issues Relative to Arithmetic for the Gifted," "A Preliminary Report on Experiments in Teaching Arithmetic in the Primary Grades," "Possible Misuses of Multisensory Aids," and "Providing Mental Arithmetic Experiences."

Dr. Fred Weaver, who is chairman of the Council's Committee on the Elementary School Curriculum in Mathematics,

is organizing one section to report the activities of that committee.

Several interesting papers on the problems of mathematics teaching at the junior high school level will also be presented. Dr. M. L. Keedy, of the University of Maryland, will tell of the "University of Maryland Mathematics Project (Junior High School)." Papers will also be presented on "Algebra in the Eighth Grade," "Issues in Junior High School Mathematics," "Set Your Sights Above the Average," and "Preparation for Algebra."

At one section, reports will be given on the activities of government and professional groups on the improvement of mathematics teaching. At this section Dr. John Mayor of the A.A.A.S. and Dr. E. G. Begle of the Yale University School Mathematics Study Group will present reports of the work of these groups.

Other sections will be on the topics "Calculus in the High School," "Probability and Statistics for High Schools," "What Algebra Should Be Taught in High Schools?" "What Geometry Should be Taught in High Schools?" Research sections and teacher education sections will also be held.

The publishing companies and other commercial organizations will exhibit the latest in textbooks, teaching aids, and equipment. The film committee is planning a showing of the latest films for use in mathematics teaching, and exhibits of mathematics projects by students from various schools will be made.

New York City mathematics teachers have long desired to entertain the members of the National Council at a meeting, and they are working hard to make this a profitable and pleasant meeting. Transportation to the world's greatest city is convenient, so the attendance should be large. If you like to combine business with pleasure, you may also wish to get a ticket to a Broadway show, take a guided tour, ride the Staten Island ferry boat (the world's biggest nickel's worth), or plan to see Time's Square on New Year's Eve.

These are exciting days for mathematics teachers. Rapid strides forward are being made, and excellent ideas are being presented with great frequency. You can ill-afford to miss so important a professional meeting if you want to keep abreast of the rapidly changing events in the world of mathematics teaching.

R. E. Pingry, University of Illinois
Program Chairman

Report of 1958 election of officers

The official count of the results of the election for officers and directors of the Council has been completed by the Remington-Rand Corporation. A report of the count has been transmitted to the president of the Council, to the Chairman of the Committee on Nominations and Elections, to the Executive Secretary, and to the Board of the National Council.

The official count shows that the following persons were elected:

President: Harold P. Fawcett
Vice-President for Senior High School: Ida May Bernhard
Vice-President for Elementary School: E. Glenadine Gibb
Directors:
Frank B. Allen, Central Region
Burton W. Jones, Southwest Region
Bruce E. Meserve, Northeast Region

It should be explained that in many cases the difference in votes was small. All candidates received substantial numbers of votes. A total of 3,675 ballots was cast.

Respectfully submitted,
Clifford Bell, Chairman,
Committee on Nominations and Elections

Report of the Membership Committee

Mary C. Rogers, Chairman, Membership Committee, Edison Junior High School, Westfield, New Jersey

This has been a remarkably successful year in National Council membership growth. Needless to say, we are very happy about it. We are confident that all of you, our current members, share this feeling with us.

This reaction on your part was very evident last April at the Annual Meeting in Cleveland. It was your keen enthusiasm about the rapid growth of the Council in numbers and in practical services; your repeated remarks about "National Council's skyrocketing membership"; your offers of direct assistance toward further growth which convinced us that our former goals, although exceptionally high, had not been too ambitious and that we should raise our sights still higher in our plans for future achievements.

There seemed to be an almost universal opinion at the Convention that the 25,000 total membership goal, set for us by Dr. Howard F. Fehr and the National Council Board in April 1957, could and should be reached by May 1, 1960 rather than by May 1, 1962 as originally planned. In light of past achievement, this increase seems reasonable and quite possible to attain.

Our membership total as of May 1, 1958 was 19,497: an increase of 3,316 during the previous year. To achieve 25,000 total membership by May 1, 1960 will necessitate an average annual increase of about 2,750 members.

True, there is the matter of increased dues for all but the Junior Members. You already have been informed of these increases in the January 1958 issues of The Mathematics Teacher and *The Arithmetic Teacher* and through direct communication from our Washington Office. We do not believe these increased dues will have any marked effect on membership maintenance and membership growth. Such has been the expressed opinion of many mathematics teachers throughout the country.

With current stress being laid on the *improvement of mathematics education at all levels;* with the *greatly enhanced interest of the general public in such improvement;* with the urgency for change in the content and point of view in keeping with modern mathematics *already being felt in our own classroom situations*, it naturally follows that the great majority of mathematics teachers feel a strong need for expert advice. In increasing numbers, they are looking to the National Council for this assistance. Let us seize every opportunity to interest *these teachers and their administrators* in the fine services which the Council has to offer.

Record of membership growth

In making our report to you, our committee has decided to speak of the goals as the "17,000 goal," the "19,000 goal," the "21,000 goal" rather than designating any specific year for reaching these goals. We recognize May 1, 1960, however, as the ultimate date for reaching the 25,000 total. Although most states are well beyond the "17,000 goal" and the "19,000 goal," we have included them in the analysis because the April 1958 goal set in April 1957 was 17,000.

In this report, our analyses use the "21,000 goal" as a maximum for comparative purposes at this time. It seems quite reasonable, however, that this maximum should be surpassed by May 1959 if we are to arrive at the 25,000 total by May 1960.

The present over-all picture shows we have already reached 93% of the 21,000 total; 31% of our states and territories have now gone beyond their quotas in this total; 23% of our states have attained 90%–99% of such goals.

This is an excellent beginning for the new year. We suggest that those who have not reached their "21,000 quotas" work toward these quotas in the near future, while the others aim at a higher goal. Later on this year, you will be informed of a suggested higher goal for all states. Does this meet with your approval?

Based on the 17,000 goal originally planned for April 1, 1958, 75% of all states and territories had already reached their goals or gone beyond them by May 1, 1958. These states and territories are:

1.	Arizona	299%
2.	Oregon	177%
3.	Nevada	167%
4.	California	145%
5.	Montana	143%
6.	Canada	141%
7.	Idaho	135%
8.	Florida	133%
9.	Michigan	132%
10.	South Dakota	132%
11.	Connecticut	131%
12.	Utah	130%
13.	New Hampshire	129%
14.	Wyoming	129%
15.	New York	126%
16.	Washington	125%
17.	Oklahoma	123%
18.	Pennsylvania	122%
19.	Maine	121%
20.	Colorado	119%
21.	New Jersey	117%
22.	Texas	117%
23.	Ohio	116%
24.	Louisiana	114%
25.	Maryland	113%
26.	Illinois	112%
27.	Kentucky	112%
28.	Mississippi	112%
29.	Hawaii & U. S. Poss.	111%
30.	Rhode Island	108%
31.	Georgia	107%
32.	Kansas	107%
33.	Massachusetts	107%
34.	Delaware	103%
35.	Minnesota	102%
36.	Foreign	102%
37.	District of Columbia	101%
38.	Iowa	101%
39.	New Mexico	101%

Based on this same 17,000 goal, membership achievement in 20% of all states shows 85%–99% of established goals:

1.	Virginia	99%
2.	Wisconsin	99%
3.	Missouri	94%
4.	Alabama	92%
5.	Vermont	91%
6.	Indiana	89%
7.	Arkansas	88%
8.	Nebraska	88%
9.	North Dakota	86%
10.	Tennessee	85%

States and Territories Having Reached Their Goals or Gone Beyond Them
(Based on the 19,000 goal)

1.	Arizona	267%
2.	Oregon	158%
3.	Nevada	147%
4.	California	130%
5.	Montana	128%
6.	Canada	126%
7.	Florida	119%
8.	Michigan	118%
9.	Connecticut	117%
10.	Idaho	117%
11.	South Dakota	117%
12.	New Hampshire	115%
13.	Utah	115%
14.	Wyoming	115%
15.	New York	113%
16.	Washington	112%
17.	Oklahoma	109%
18.	Pennsylvania	109%
19.	Maine	108%
20.	Colorado	107%
21.	New Jersey	105%
22.	Texas	105%
23.	Ohio	104%
24.	Louisiana	102%
25.	Maryland	101%
26.	Illinois	100%
27.	Kentucky	100%

Membership Achievement 85%–99%
(Based on the 19,000 goal)

1.	Mississippi	99%
2.	Hawaii & U. S. Poss.	99%
3.	Rhode Island	97%
4.	Georgia	96%
5.	Kansas	96%
6.	Massachusetts	96%
7.	Delaware	92%
8.	Minnesota	92%
9.	Foreign	91%
10.	District of Columbia	90%
11.	Iowa	90%
12.	New Mexico	90%
13.	Wisconsin	88%
14.	Virginia	87%

States and Territories Having Reached Their Goals or Gone Beyond Them
(Based on the 21,000 goal)

1.	Arizona	242%
2.	Oregon	143%
3.	Nevada	132%
4.	California	117%
5.	Montana	115%
6.	Canada	114%
7.	Florida	108%
8.	Idaho	108%
9.	South Dakota	108%
10.	Michigan	107%
11.	Connecticut	106%
12.	Wyoming	106%
13.	New Hampshire	104%
14.	Utah	104%
15.	New York	102%
16.	Washington	101%

Membership Achievement 85%–99%
(Based on the 21,000 goal)

1.	Pennsylvania	99%
2.	Maine	97%
3.	Oklahoma	97%
4.	Colorado	96%
5.	New Jersey	95%
6.	Texas	95%
7.	Ohio	93%
8.	Louisiana	92%
9.	Illinois	91%
10.	Kentucky	91%
11.	Maryland	91%
12.	Mississippi	90%
13.	Hawaii & U. S. Poss.	89%
14.	Georgia	87%
15.	Kansas	87%
16.	Massachusetts	87%
17.	Rhode Island	87%

Leaders in Membership Totals
(Including subscriptions)

1.	New York	1686
2.	Illinois	1513
3.	California	1445
4.	Pennsylvania	1271
5.	Ohio	972
6.	Michigan	900
7.	Texas	890
8.	New Jersey	743
9.	Indiana	544
10.	Massachusetts	535
11.	Florida	528
12.	Wisconsin	470
13.	Minnesota	465
14.	Virginia	402
15.	Foreign	391

Leaders in Membership Totals
(Not including subscriptions)

1.	New York	1200
2.	Illinois	1177
3.	Pennsylvania	973
4.	California	825
5.	Ohio	800
6.	Michigan	657
7.	Texas	631
8.	New Jersey	534
9.	Indiana	454
10.	Florida	418
11.	Massachusetts	409
12.	Wisconsin	327
13.	Minnesota	322
14.	Virginia	312
15.	Kansas	308

Leaders in Membership Growth
(Including subscriptions)

1.	Illinois	370
2.	New York	308
3.	Michigan	260
4.	Ohio	225
5.	California	182
6.	Pennsylvania	178
7.	New Jersey	145
8.	Texas	112
9.	Florida	105
10.	Minnesota	97
11.	Massachusetts	93
12.	Oklahoma	82
13.	Connecticut	80
14.	Kansas	78
15.	Louisiana	77

States and Territories with Greatest Per Cent of Growth Since May 1, 1957

1. Wyoming	9. Ohio
2. Rhode Island	10. Idaho
3. Montana	11. Louisiana
4. Michigan	12. Kansas
5. Kentucky	13. Minnesota
6. Oklahoma	14. West Virginia
7. Illinois	15. Florida
8. Connecticut	16. New Jersey

States and Territories with Continuous Growth
(Since May 1, 1957)

1. Alabama	9. Missouri
2. California	10. Nebraska
3. Delaware	11. New Hampshire
4. Iowa	12. North Dakota
5. Kentucky	13. Oklahoma
6. Louisiana	14. Texas
7. Massachusetts	15. Washington
8. Michigan	16. Wyoming

The National Council of Teachers of Mathematics Analysis of Membership Growth, Members and Subscribers, May 1957 to May 1958

		May 1958		*Goals and Per Cents of Goals Reached*					
				17,000		*19,000*		*21,000*	
	May 1957	*Individuals*	*Totals*	*Goal*	*%*	*Goal*	*%*	*Goal*	*%*
Alabama	147	133	177	192	92%	215	82%	237	75%
Arizona	165	139	203	68	299%	76	267%	84	242%
Arkansas	156	121	170	194	88%	217	78%	239	71%
California	1,163	825	1,445	996	145%	1,113	130%	1,231	117%
Colorado	208	194	243	204	119%	228	107%	252	96%
Connecticut	259	225	339	258	131%	289	117%	319	106%
Delaware	69	63	82	80	103%	89	92%	99	83%
Dist. of Columbia	198	188	211	209	101%	234	90%	258	82%
Florida	423	418	528	398	133%	445	119%	491	108%
Georgia	185	170	226	211	107%	236	96%	260	87%
Idaho	21	17	27	20	135%	23	117%	25	108%
Illinois	1,143	1,177	1,513	1,346	112%	1,506	100%	1,663	91%
Indiana	557	454	544	614	89%	686	79%	758	72%
Iowa	297	247	338	335	101%	374	90%	414	82%
Kansas	299	308	377	352	107%	393	96%	435	87%
Kentucky	115	126	156	139	112%	156	100%	172	91%
Louisiana	279	254	356	313	114%	350	102%	386	92%
Maine	78	77	94	78	121%	87	108%	97	97%
Maryland	301	276	365	323	113%	361	101%	399	91%
Massachusetts	442	409	535	498	107%	557	96%	615	87%
Michigan	640	657	900	680	132%	760	118%	840	107%
Minnesota	368	322	465	454	102%	507	92%	561	83%
Mississippi	137	116	155	139	112%	156	99%	172	90%
Missouri	316	260	350	374	94%	418	84%	462	76%
Montana	68	77	97	68	143%	76	128%	84	115%
Nebraska	142	124	164	187	88%	209	78%	231	71%
Nevada	23	16	25	15	167%	17	147%	19	132%
New Hampshire	82	77	94	73	129%	82	115%	90	104%
New Jersey	598	534	743	636	117%	711	105%	785	95%
New Mexico	86	73	101	100	101%	112	90%	124	81%
New York	1,378	1,200	1,686	1,338	126%	1,495	113%	1,653	102%
North Carolina	209	172	237	289	82%	323	73%	357	67%
North Dakota	35	29	42	49	86%	55	76%	61	69%
Ohio	747	800	972	838	116%	937	104%	1,035	93%
Oklahoma	245	255	327	265	123%	300	109%	332	97%
Oregon	230	173	267	151	177%	169	158%	187	143%
Pennsylvania	1,093	973	1,271	1,042	122%	1,165	109%	1,287	99%
Rhode Island	55	60	86	80	108%	89	97%	99	87%
South Carolina	105	77	127	155	82%	173	73%	191	67%
South Dakota	56	33	54	41	132%	46	117%	50	108%
Tennessee	213	178	236	279	85%	312	76%	344	69%
Texas	778	631	890	762	117%	851	105%	941	95%
Utah	64	44	70	54	130%	61	115%	67	104%
Vermont	38	33	42	46	91%	51	82%	57	74%
Virginia	352	312	402	405	99%	452	87%	500	80%
Washington	270	194	323	258	125%	289	112%	319	101%
West Virginia	91	92	115	179	64%	200	58%	220	52%
Wisconsin	470	327	470	476	99%	532	88%	588	80%
Wyoming	33	37	53	41	129%	46	115%	50	106%
Totals	15,427	13,697	18,693	16,305	115%	18,228	103%	20,140	93%
Hawaii & U. S. Poss.	89	61	92	83	111%	93	99%	103	89%
Canada	297	189	321	228	141%	255	126%	282	114%
Foreign	368	183	391	384	102%	429	91%	475	82%
Grand Totals	16,181	14,130	19,497	17,000	115%	19,005	103%	21,000	93%

Future plans

Seldom in the history of the National Council has there been such widespread interest in mathematics education at all levels of instruction. Let us respond to this heightened interest by stepping up our publicity and publicity releases in keeping with the times.

1. You, the current members of the Council, are our most effective publicity agents. It has been your outstanding co-operation in our "Each-One-Win-One" procedures of the past which has been largely responsible for our most commendable membership growth. We are counting on you for a continuance and expansion of this fine service. If we can help you in any way by supplying you with added information and materials with which to work, please write us about it.
2. We cannot emphasize too strongly the importance of National Council publications, The Mathematics Teacher, *The Arithmetic Teacher*, and *The Mathematics Student Journal*, the supplementary publications, the publications of results of research in mathematics education, and the yearbooks. Current and invaluable is the *Twenty-third Yearbook, Insights into Modern Mathematics.* Attention should be called to the fact that special prices are given to Council members on all Council yearbooks.
3. A recent letter from our president, Dr. Harold P. Fawcett, informs us of the following expanded Council services.
   a. "The Committee on Supplementary Publications will, in course of time, make available appropriate materials on modern mathematics which should be most helpful to our membership.
   b. "The Secondary School Curriculum Committee is in process of preparing an interim report which will be designed to answer the many, many questions which have been received by our National office.
   c. "I am also anticipating that the Committee on Teacher Education, Certification, and Recruitment, as well as the Research Committee, will develop materials of value."
4. Additional publicity materials regarding the expanding services of the Council are being prepared this summer by Mr. M. H. Ahrendt and will be available to you early next fall.
5. National Council conventions are invaluable in the inspiration and direct assistance which they afford to all who can possibly attend them. We shall look forward to seeing you at these meetings. Do come and bring several of your colleagues with you.
6. Members of the mathematics staff in state teachers colleges and other great schools of education are doing the Council a tremendous service by interesting their undergraduate students in junior membership in the Council. May we remind you:
   a. Students qualify for junior membership throughout the *four years of their undergraduate study*, provided their status as students is endorsed each year with the National Council Washington office by a staff member of the institution which they attend.
   b. The student membership fee *remains the same* as formerly, with one journal available for **$1.50** a year, and both for **$2.50**.
7. You who are supervisors and/or mathematics department chairmen are obtaining increasingly fine results in your work with your teachers. We are counting on your continued help in interesting these teachers in National Council services.
8. State and local associations of mathematics teachers, in co-operation with State Representatives to the National Council, are performing an outstanding service to the Council. A very great deal of the strength of the Council is due to your loyalty and your

enthusiastic support. A continuance of your fine work is vital to the success of Council services.

9. Library and other institutional subcriptions will continue to be included in preparing reports of membership growth. Check with your school to see that these subscriptions are renewed.
10. Prompt renewal of membership is always greatly appreciated. This greatly facilitates the keeping of accurate records and the preparation of reports which are right up to the minute.

The Membership Committee thanks each of you most sincerely for your enthusiatic co-operation and support at all times. Please accept our best wishes for a most successful professional year just ahead.

PEARL BOND
MARIAN C. CLIFFE
MARY LEE FOSTER
JANET HEIGHT
LUCILLE HOUSTON
HAROLD J. HUNT
FLORENCE INGHAM
MILDRED KEIFFER
FAITH NOVINGER
BESS PATTON

M. H. AHRENDT and
ELIZABETH ROUDEBUSH,
Ex Officio members

MARY C. ROGERS, Chairman

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N. W., Washington 6, D. C.

NCTM convention dates

EIGHTEENTH CHRISTMAS MEETING

December 29–30, 1958
Sheraton-McAlpin Hotel, New York, New York
Elizabeth Sibley, 18 Stuyvesant Oval, New York 9, New York

THIRTY-SEVENTH ANNUAL MEETING

April 1–4, 1959
Baker Hotel, Dallas, Texas
Arthur W. Harris, 4701 Cole Avenue, Dallas 5, Texas

JOINT MEETING WITH NEA

June 29, 1959
St. Louis, Missouri
M. H. Ahrendt, 1201 Sixteenth Street, N. W. Washington 6, D. C.

NINETEENTH SUMMER MEETING

August 17–19, 1959
University of Michigan, Ann Arbor, Michigan
Phillip S. Jones, Mathematics Department, University of Michigan, Ann Arbor, Michigan

AN INTRODUCTION TO SETS AND THE STRUCTURE OF ALGEBRA

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and HELEN R. PEARSON
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